Topp-Leone Exponential Distribution For Asymmetric Loss Functions With Identical Priors

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ABSTRACT

The present paper considers estimating the shape and scale parameters of the Topp-Leone Exponential distribution. Bayes estimators are obtained using exponential, gamma, log-normal, and Weibull distributions as the identical priors under asymmetric loss functions and integrated with the Lindley approximation method. These priors are compared using Bayes risk through simulation study with varying sample sizes and real data sets. Specifically, for the shape parameter of the Topp-Leone exponential distribution, the study identifies that Gamma prior under the Entropy loss function is most preferred.

Keywords: *Prior, Lindley's approximation, Asymmetric loss functions, Bayes Estimator, Bayes Risk*

JEL Classification: *C11*

1. INTRODUCTION

Bayesian estimation, a non-classical approach to statistical inference, is widely applied globally. The Topp-Leone distribution, a bounded J-shaped distribution, is an alternative to the Beta distribution. Various authors have studied this distribution. The Exponential distribution proposed by Epstein (1954) plays a significant role in real-life scenarios. Topp and Leone (1955) proposed that the Topp-Leone distribution includes discussions on its bounded variant, for analyzing empirical data characterized by J-shaped histograms. Nadarajah and Kotz (2003) determined some J-shaped distribution moments of Topp-Leone distribution. Kotz and Seier (2007) compared the kurtosis of the Topp-Leone and left triangular distributions. Genc (2012) presented recurrence relations for the moments of order statistics from the Topp–Leone distribution. Al-Shomrani et al (2016) introduced the Topp-Leone family of distributions, providing a comprehensive overview of its characteristics and practical applications. Mohammed et al (2018) studied the comparison of the Topp-Leone Exponential, Topp-Leone Exponentiated exponential and Topp-Leone Exponentiated expansion models for ovarian cancer patient data. Fatoki Olayode (2019) discussed the moment generation function, survival function and ordinal statistics of the Topp-Leone Rayleigh distribution. Kawsar et al (2017) estimated the shape parameter of Exponentiated moment Exponential distributions using informative and non-informative priors under the SELF, PLF and Entropy loss functions. Hind Jawad Kadhim Albderi (2021) discussed the survival function of the Topp-Leone exponential distribution and its application. Noman Rashed (2019) studied the properties and applications of the Topp-Leone compound Rayleigh distribution. Radha et al (2017) discussed the classical and Bayesian estimation of Power function distribution. Randhir Singh (2021) investigated Bayesian parameter estimation of the Exponential distribution using type II censored samples, employing various loss functions such as Squared Error, DeGroot, Minimum expected loss, and Exponentially weighted minimum expected loss. Fithriani et al (2019) utilized SELF and PLF to estimate the shape parameter k of the Burr distribution, by comparing both symmetric and asymmetric loss functions. Sanjay Kumar Singh et al (2011) discussed parameter estimation of the Exponentiated Exponential distribution and its reliability function for type II censored data using the entropy loss function. Farouk et al (2019) focused on parameter estimation of the Lindley distribution using informative and non-informative priors under the Linex loss function. Saridha et al (2024) discussed the Topp-Leone Exponential distribution, emphasizing the role of symmetric loss functions with identical priors. This paper adopts the Bayesian approach for Topp-Leone Exponential Distribution to estimate the parameters with identical priors using asymmetric loss functions. The unknown shape and scale parameter are assumed to follow identical priors presented in Table:1 for Topp -Leone Exponential distribution.

Priors Selection

Table 1

Priors	***Identical priors***	
	Shape Parameter $\boldsymbol{\eta}$	*Scale Parameter* $\boldsymbol{\delta}$
Exponential	*Exponential*	*Exponential*
Gamma	*Gamma*	*Gamma*
Log-Normal	*Log-Normal*	*Log-Normal*
Weibull	*Weibull*	*Weibull*

2. MAXIMUM LIKELIHOOD ESTIMATION

The p.d.f of Topp-Leone Exponential distribution (Al-Shomrani et al,2016) is given by

$$f(x;\eta,\delta) = 2\eta\delta\ e^{-2\delta x}\left(1 - e^{-2\delta x}\right)^{\eta-1}\ ; x,\eta,\delta \geq 0 \qquad [1]$$

with η as shape parameter and δ the scale parameter.

Then the likelihood function:

$$L(x;\eta,\delta) = (2\eta\delta)^n\ e^{-2\delta\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}\left(1 - e^{-2\delta x_i}\right)^{\eta-1} \qquad [2]$$

Taking the log of likelihood equation [2] and differentiating $w.r.to.\eta$ and δ gives

$$\frac{\partial LogL}{\partial\eta} = \frac{n}{\eta} + \sum_{i=1}^{n} log\left(1 - e^{-2\delta x_i}\right) = 0 \qquad [3]$$

$$\frac{\partial LogL}{\partial\delta} = \frac{n}{\delta} - 2\sum_{i=1}^{n} x_i + (\eta - 1)\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1 - e^{-2\delta x_i})} = 0 \qquad [4]$$

The maximum likelihood estimates (MLEs) of η and δ, say $\hat{\eta}$ and $\hat{\delta}$, respectively, are the solution of the equations [3] and [4]. Unfortunately, analytic solutions for η and δ are not in the closed form. To estimate these parameters η and δ Newton Raphson's method is used..

3. PRIORS AND POSTERIOR DISTRIBUTIONS

In Topp-Leone Exponential distribution, it is assumed that the shape parameter η and scale parameter δ both have identical priors namely, Exponential(E) - Exponential(E), Gamma (G)- Gamma (G), Log Normal (LN)-Log Normal (LN) and Weibull(W)-Weibull (W). The posterior distribution with identical priors for the shape and scale parameters are discussed as follows:

3.1. Posterior Distribution for Topp-Leone Exponential distribution using Identical Priors

3.1.1 Exponential Prior

The joint prior distribution using Exponential priors for both η and δ $i.e., \eta \sim exp(a_1)$ and $\delta \sim exp(a_2)$ is :

$$p_1(\eta,\delta) = a_1 a_2 e^{-(a_1\eta + a_2\delta)}\ ; a_1, a_2 > 0 \qquad [5]$$

$$\pi_1(\eta,\delta|x) = \frac{1}{C_1} e^{-(a_1\eta+a_2\delta)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i} \prod_{i=1}^{n} \left(1 - e^{-2\delta x_i}\right)^{\eta-1} \qquad [6]$$

where

$$C_1 = \int_0^\infty \int_0^\infty e^{-(a_1\eta+a_2\delta)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i} \prod_{i=1}^{n} \left(1 - e^{-2\delta x_i}\right)^{\eta-1} d\eta d\delta$$

3.1.2. Gamma Prior

The joint prior distribution using Gamma priors for both η and δ $i.e., \eta \sim G(a_3, b_1)$ and $\delta \sim G(a_4, b_2)$ is:

$$p_2(\eta,\delta) = \frac{b_1 b_2}{\Gamma a_3 \Gamma a_4} \eta^{a_3-1}\delta^{a_4-1} e^{-(b_1\eta+b_2\delta)};\ \eta, \delta, a_3, a_4, b_1, b_2 > 0 \qquad [7]$$

The joint posterior distribution of η and δ is given by:

$$\pi_2(\eta,\delta|x) = \frac{1}{C_2} \eta^{a_3-1}\delta^{a_4-1} e^{-(b_1\eta+b_2\delta)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i} \prod_{i=1}^{n} \left(1 - e^{-2\delta x_i}\right)^{\eta-1} \qquad [8]$$

where

$$C_2 = \int_0^\infty \int_0^\infty \eta^{a_3-1}\delta^{a_4-1} e^{-(b_1\eta+b_2\delta)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i} \prod_{i=1}^{n} \left(1 - e^{-2\delta x_i}\right)^{\eta-1} d\eta d\delta$$

3.1.3. Log-Normal Prior

The joint prior distribution using Log-Normal prior for both η and δ $i.e., \eta \sim LN(a_5, b_3)$ and $\delta \sim LN(a_6, b_4)$

$$p_3(\eta,\delta) = \frac{1}{\eta b_3 \sqrt{2\pi_1}} e^{-\frac{(log\eta - a_5)^2}{2b_3^2}} \frac{1}{\delta b_4 \sqrt{2\pi_2}} e^{-\frac{(log\delta - a_6)^2}{2b_4^2}}, a_5, a_6, b_3, b_4 > 0 \qquad [9]$$

The joint posterior distribution of η and δ is given by:

$$\pi_3(\eta,\delta|x) = \frac{1}{C_3}\frac{1}{\eta\delta} e^{-\frac{(log\eta - a_5)^2}{2b_3^2}} e^{-\frac{(log\delta - a_6)^2}{2b_4^2}} (2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i} \prod_{i=1}^{n} \left(1 - e^{-2\delta x_i}\right)^{\eta-1} d\eta d\delta \qquad [10]$$

where

$$C_3 = \int_0^\infty \int_0^\infty \frac{1}{\eta\delta} e^{-\frac{(log\eta - a_5)^2}{2b_3^2}} e^{-\frac{(log\delta - a_6)^2}{2b_4^2}} (2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i} \prod_{i=1}^{n} \left(1 - e^{-2\delta x_i}\right)^{\eta-1} d\eta d\delta$$

3.1.4. Weibull Prior

The joint prior distribution using Weibull prior for both η and δ $i.e.,\eta\sim W(a_7,b_5)$ and $\delta\sim W(a_8,b_6)$ is:

$$p_4(\eta,\delta)=\frac{a_7}{b_5^{a_7}}\eta^{a_7-1}e^{-\left(\frac{\eta}{b_5}\right)^{a_7}}\frac{a_8}{b_6^{a_8}}\delta^{a_8-1}e^{-\left(\frac{\delta}{b_6}\right)^{a_8}} \quad [11]$$

The joint posterior distribution of η and δ is given by:

$$\pi_4(\eta,\delta|x)=\frac{1}{c_4}\eta^{a_7-1}e^{-\left(\frac{\eta}{b_5}\right)^{a_7}}\delta^{a_8-1}e^{-\left(\frac{\delta}{b_6}\right)^{a_8}}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i}\prod_{i=1}^n\left(1-e^{-2\delta x_i}\right)^{\eta-1}d\eta d\delta \quad [12]$$

where

$$C_4=\int_0^\infty\int_0^\infty\eta^{a_7-1}e^{-\left(\frac{\eta}{b_5}\right)^{a_7}}\delta^{a_8-1}e^{-\left(\frac{\delta}{b_6}\right)^{a_8}}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i}\prod_{i=1}^n\left(1-e^{-2\delta x_i}\right)^{\eta-1}d\eta d\delta$$

4. BAYES ESTIMATES UNDER DIFFERENT LOSS FUNCTIONS

To estimate the parameters of Topp-Leone Exponential distribution for asymmetric loss functions namely, DeGroot, Linear Exponential loss function (LINEX) and General Entropy loss function presented in TABLE: 2 are considered.

Bayes estimators and Bayes risk for various loss functions

Table 2

Loss Function Expression	***Bayes Estimator***		***Bayes risk***	
	Parameter η	Parameter δ	Parameter η	Parameter δ
DEGROOT $=L((\hat{\eta}-\eta)\propto\left(\frac{\hat{\eta}-\eta}{\hat{\eta}}\right)^2$	$\hat{\eta}_{DEGROOT}=\frac{E(\eta^2\|x)}{E(\eta\|x)}$	$\hat{\delta}_{DEGROOT}=\frac{E(\delta^2\|x)}{E(\delta\|x)}$	$\frac{Var(\eta\|x)}{E(\eta^2\|x)}$	$\frac{Var(\delta\|x)}{E(\delta^2\|x)}$
$LINEX=L((\hat{\eta}-\eta)\propto\exp(a_1(\hat{\eta}-\eta))-a_1(\hat{\eta}-\eta)-1$	$\hat{\eta}_{LINEX}=-\frac{1}{m}\log[E(e^{-m\eta x\|x})]$	$\hat{\delta}_{LINEX}=-\frac{1}{m}\log[E(e^{-m\delta x\|x})]$	$\log[E(e^{-m\eta x\|x})]+mE(\eta\|x)$	$\log[E(e^{-m\delta x\|x})]+mE(\delta\|x)$
$ENTROPY=L((\hat{\eta}-\eta)\propto\left(\frac{\hat{\eta}}{\eta}\right)-a\log\left(\frac{\hat{\eta}}{\eta}\right)-1$	$\hat{\eta}_{ENTROPY}=[E(\eta^{-c}\|x]^{-\frac{1}{c}}$	$\hat{\delta}_{ENTROPY}=[E(\delta^{-c}\|x]^{-\frac{1}{c}}$	$cE(log\eta\|x)-clog[E(\eta^{-c}\|x]^{-\frac{1}{c}}$	$cE(log\delta\|x-clog[E(\delta^{-c}\|x]^{-\frac{1}{c}}$

The joint posterior distribution given in the equations [6], [8], [10] and [12] cannot be solved analytically to estimate the parameters η and δ. Hence, the Lindley approximation method is adopted. The posterior expectation can be expressed as (Anitta et al,2020).

$$E[u(\eta,\delta)|x]=\frac{\int u(\eta,\delta)\exp[L(\eta,\delta)+\rho(\eta,\delta)]\,d(\eta,\delta)}{\int\exp[L(\eta,\delta)+\rho(\eta,\delta)]\,d(\eta,\delta)} \quad [13]$$

where $u(\eta,\delta)$ is a function of η and δ only, $L(\eta,\delta)$ is the log-likelihood and $\rho(\eta,\delta)$ is the log of the joint prior of η and δ.

According to Lindley (1980), if the sample size nn is sufficiently large, the above equation can be approximately evaluated through:

$$I(x) = u(\hat{\eta},\hat{\delta}) + \frac{1}{2}[(u_{\eta\eta} + 2u_\eta\rho_\eta)\sigma_{\eta\eta} + (u_{\delta\eta} + 2u_\delta\rho_\eta)\sigma_{\delta\eta} + (u_{\eta\delta} + 2u_\eta\rho_\delta)\sigma_{\eta\delta}$$
$$+(u_{\delta\delta} + 2u_\delta\rho_\delta)\sigma_{\delta\delta} + \frac{1}{2}[(u_\eta\sigma_{\eta\eta} + u_\delta\sigma_{\eta\delta})(L_{\eta\eta\eta}\sigma_{\eta\eta} + L_{\eta\delta\eta}\sigma_{\eta\delta} + L_{\delta\eta\eta}\sigma_{\delta\eta} + L_{\delta\delta\eta}\sigma_{\delta\delta})]$$
$$+\frac{1}{2}[(u_\eta\sigma_{\delta\eta} + u_\delta\sigma_{\delta\delta})(L_{\eta\eta\delta}\sigma_{\eta\eta} + L_{\eta\delta\delta}\sigma_{\eta\delta} + L_{\delta\eta\delta}\sigma_{\delta\eta} + L_{\delta\delta\delta}\sigma_{\delta\delta})] \quad [14]$$

$u_\eta = \frac{\partial u(\eta,\delta)}{\partial \eta}$; $u_\delta = \frac{\partial u(\eta,\delta)}{\partial \delta}$; $u_{\eta\eta} = \frac{\partial^2 u(\eta,\delta)}{\partial \eta^2}$; $u_{\delta\delta} = \frac{\partial^2 u(\eta,\delta)}{\partial \delta^2}$; $u_{\eta\delta} = \frac{\partial^2 u(\eta,\delta)}{\partial \eta \partial \delta}$; $\frac{\partial^2 LogL}{\partial \eta^2} = L_{\eta\eta}$ and so on.

with the above-defined expressions, the values of the estimates for the Topp-Leone Exponential distribution are as follows.

$$E[u(\hat{\eta},\hat{\delta})|x] = u(\hat{\eta},\hat{\delta}) + \frac{1}{2}[(u_{\eta\eta} + 2u_\eta\rho_\eta)\sigma_{\eta\eta} + (u_{\delta\eta} + 2u_\delta\rho_\eta)\sigma_{\delta\eta} + (u_{\eta\delta} + 2u_\eta\rho_\delta)\sigma_{\eta\delta}$$
$$+(u_{\delta\delta} + 2u_\delta\rho_\delta)\sigma_{\delta\delta} + \frac{1}{2}[(u_\eta\sigma_{\eta\eta} + u_\delta\sigma_{\eta\delta})(S_1)] + \frac{1}{2}[(u_\eta\sigma_{\delta\eta} + u_\delta\sigma_{\delta\delta})(S_2)] \quad [15]$$

Where $S_1 = L_{\eta\eta\eta}\sigma_{\eta\eta} + L_{\delta\delta\eta}\sigma_{\delta\delta} S_2 = L_{\eta\delta\delta}\sigma_{\eta\delta} + L_{\delta\eta\delta}\sigma_{\delta\eta} + L_{\delta\delta\delta}\sigma_{\delta\delta}$

Then the logarithmic joint prior density of:

(i) Exponential prior :

$$\rho(\eta,\delta) = loga_1 + loga_2 - a_1\eta - a_2\delta$$

$$\rho(\eta) = \frac{\partial \rho(\eta,\delta)}{\partial \eta} = -a_1 \quad [16]$$

$$\rho(\delta) = \frac{\partial \rho(\eta,\delta)}{\partial \delta} = -a_2 \quad [17]$$

(ii) Gamma prior :

$$\rho(\eta,\delta) = (a_3 - 1)log\eta - b_1\eta + (a_4 - 1)log\delta - b_2\delta$$

$$\rho(\eta) = \frac{\partial \rho(\eta,\delta)}{\partial \eta} = \frac{a_3 - 1}{\eta} - b_1 \quad [18]$$

$$\rho(\delta) = \frac{\partial \rho(\eta,\delta)}{\partial \delta} = \frac{a_4 - 1}{\delta} - b_2 \quad [19]$$

(iii) Log-Normal prior :

$$\rho(\eta,\delta) = \log\left(\frac{1}{\eta\delta}\right) - \frac{(log\eta - a_5)^2}{2b_3^2} - \frac{(log\delta - a_6)^2}{2b_4^2}$$

$$\rho(\eta) = \frac{\partial \rho(\eta, \delta)}{\partial \eta} = -\frac{1}{\eta} - \frac{\log\eta - a_5}{\eta b_3^2} \quad [20]$$

$$\rho(\delta) = \frac{\partial \rho(\eta, \delta)}{\partial \delta} = -\frac{1}{\delta} - \frac{\log\delta - a_6}{\delta b_4^2} \quad [21]$$

(iv) Weibull prior :

$$\rho(\eta, \delta) = (a_7 - 1)\log\eta + (a_8 - 1)\log\delta - \left(\frac{\eta}{b_5}\right)^{a_7} - \left(\frac{\delta}{b_6}\right)^{a_8}$$

$$\rho(\eta) = \frac{\partial \rho(\eta, \delta)}{\partial \eta} = \frac{a_7 - 1}{\eta} - \frac{a_7}{b_5}\left(\frac{\eta}{b_5}\right)^{a_7 - 1} \quad [22]$$

$$\rho(\delta) = \frac{\partial \rho(\eta, \delta)}{\partial \delta} = \frac{a_8 - 1}{\delta} - \frac{a_8}{b_6}\left(\frac{\delta}{b_6}\right)^{a_8 - 1} \quad [23]$$

4.1 Lindley's Approximation of η and δ using DEGROOT:

The Bayes estimate for the parameter η of Exponential, Gamma, Log-Normal, and Weibull priors using equation [14] are given by:

$$\hat{\eta}_E = \left[\frac{\hat{\eta}^2 + \sigma_{\eta\eta} - 2\hat{\eta}(a_1)\sigma_{\eta\eta} - 2\hat{\eta}(a_2)\sigma_{\eta\delta} + \hat{\eta}\sigma_{\eta\eta}S_1 + \hat{\eta}\sigma_{\delta\eta}S_2}{\hat{\eta} - (a_1)\sigma_{\eta\eta} - (a_2)\sigma_{\eta\delta} + \frac{1}{2}\sigma_{\eta\eta}S_1 + \frac{1}{2}\sigma_{\delta\eta}S_2}\right] \quad [24]$$

$$\hat{\eta}_G = \left[\frac{\hat{\eta}^2 + \sigma_{\eta\eta} + 2\hat{\eta}\left(\frac{a_3 - 1}{\hat{\eta}} - b_1\right)\sigma_{\eta\eta} + 2\hat{\eta}\left(\frac{a_4 - 1}{\hat{\delta}} - b_2\right)\sigma_{\eta\delta} + \hat{\eta}\sigma_{\eta\eta}S_1 + \hat{\eta}\sigma_{\delta\eta}S_2}{\hat{\eta} + \left(\frac{a_3 - 1}{\hat{\eta}} - b_1\right)\sigma_{\eta\eta} + \left(\frac{a_4 - 1}{\hat{\delta}} - b_2\right)\sigma_{\eta\delta} + \frac{1}{2}\sigma_{\eta\eta}S_1 + \frac{1}{2}\sigma_{\delta\eta}S_2}\right] \quad [25]$$

$$\hat{\eta}_{LN} = \left[\frac{\hat{\eta}^2 + \sigma_{\eta\eta} + 2\hat{\eta}\left(-\frac{1}{\hat{\eta}} - \frac{\log\hat{\eta} - a_5}{\hat{\eta} b_3^2}\right)\sigma_{\eta\eta} + 2\hat{\eta}\left(-\frac{1}{\hat{\delta}} - \frac{\log\hat{\delta} - a_6}{\hat{\delta} b_4^2}\right)\sigma_{\eta\delta} + \hat{\eta}\sigma_{\eta\eta}S_1 + \hat{\eta}\sigma_{\delta\eta}S_2}{\hat{\eta} + \left(-\frac{1}{\hat{\eta}} - \frac{\log\hat{\eta} - a_5}{\hat{\eta} b_3^2}\right)\sigma_{\eta\eta} + \left(-\frac{1}{\hat{\delta}} - \frac{\log\hat{\delta} - a_6}{\hat{\delta} b_4^2}\right)\sigma_{\eta\delta} + \frac{1}{2}\sigma_{\eta\eta}S_1 + \frac{1}{2}\sigma_{\delta\eta}S_2}\right] \quad [26]$$

$$\hat{\eta}_W = \left[\frac{\hat{\eta}^2 + \sigma_{\eta\eta} + 2\hat{\eta}\left(\frac{a_7 - 1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7 - 1}\right)\sigma_{\eta\eta} + 2\hat{\eta}\left(\frac{a_8 - 1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8 - 1}\right)\sigma_{\eta\delta} + \hat{\eta}\sigma_{\eta\eta}S_1 + \hat{\eta}\sigma_{\delta\eta}S_2}{\hat{\eta} + \left(\frac{a_7 - 1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7 - 1}\right)\sigma_{\eta\eta} + \left(\frac{a_8 - 1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8 - 1}\right)\sigma_{\eta\delta} + \frac{1}{2}\sigma_{\eta\eta}S_1 + \frac{1}{2}\sigma_{\delta\eta}S_2}\right] \quad [27]$$

The Bayes estimate for the parameter δ of Exponential, Gamma, Log-Normal, and Weibull priors using equation [14] are given by:

$$\hat{\delta}_E = \left[\frac{\hat{\delta}^2 + \sigma_{\delta\delta} - 2\hat{\delta}(a_1)\sigma_{\delta\eta} - 2\hat{\delta}(a_2)\sigma_{\delta\delta} + \hat{\delta}\sigma_{\eta\delta}S_1 + \hat{\delta}\sigma_{\delta\delta}S_2}{\hat{\delta} - (a_1)\sigma_{\delta\eta} - (a_2)\sigma_{\delta\delta} + \frac{1}{2}\sigma_{\eta\delta}S_1 + \frac{1}{2}\sigma_{\delta\delta}S_2}\right] \quad [28]$$

$$\hat{\delta}_G = \left[\frac{\hat{\delta}^2 + \sigma_{\delta\delta} + 2\hat{\delta}\left(\frac{a_3-1}{\hat{\eta}} - b_1\right)\sigma_{\delta\eta} + 2\hat{\delta}\left(\frac{a_4-1}{\hat{\delta}} - b_2\right)\sigma_{\delta\delta} + \hat{\delta}\sigma_{\eta\delta}S_1 + \hat{\delta}\sigma_{\delta\delta}S_2}{\hat{\delta} + \left(\frac{a_3-1}{\hat{\eta}} - b_1\right)\sigma_{\delta\eta} + \left(\frac{a_4-1}{\hat{\delta}} - b_2\right)\sigma_{\delta\delta} + \frac{1}{2}\sigma_{\eta\delta}S_1 + \frac{1}{2}\sigma_{\delta\delta}S_2}\right] \quad [29]$$

$$\hat{\delta}_{LN} = \left[\frac{\hat{\delta}^2 + \sigma_{\delta\delta} + 2\hat{\delta}\left(-\frac{1}{\hat{\eta}} - \frac{log\hat{\eta} - a_5}{\hat{\eta}b_3^2}\right)\sigma_{\delta\eta} + 2\hat{\delta}\left(-\frac{1}{\hat{\delta}} - \frac{log\hat{\delta} - a_6}{\hat{\delta}b_4^2}\right)\sigma_{\delta\delta} + \hat{\delta}\sigma_{\eta\delta}S_1 + \hat{\delta}\sigma_{\delta\delta}S_2}{\hat{\delta} + \left(-\frac{1}{\hat{\eta}} - \frac{log\hat{\eta} - a_5}{\hat{\eta}b_3^2}\right)\sigma_{\delta\eta} + \left(-\frac{1}{\hat{\delta}} - \frac{log\hat{\delta} - a_6}{\hat{\delta}b_4^2}\right)\sigma_{\delta\delta} + \frac{1}{2}\sigma_{\eta\delta}S_1 + \frac{1}{2}\sigma_{\delta\delta}S_2}\right] \quad [30]$$

$$\hat{\delta}_W = \left[\frac{\hat{\delta}^2 + \sigma_{\delta\delta} + 2\hat{\delta}\left(\frac{a_7-1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7-1}\right)\sigma_{\delta\eta} + 2\hat{\delta}\left(\frac{a_8-1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8-1}\right)\sigma_{\delta\delta} + \hat{\delta}\sigma_{\eta\delta}S_1 + \hat{\delta}\sigma_{\delta\delta}S_2}{\hat{\delta} + \left(\frac{a_7-1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7-1}\right)\sigma_{\delta\eta} + \left(\frac{a_8-1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8-1}\right)\sigma_{\delta\delta} + \frac{1}{2}\sigma_{\eta\delta}S_1 + \frac{1}{2}\sigma_{\delta\delta}S_2}\right] \quad [31]$$

4.2 Lindley's Approximation of η and δ using LINEX:

The Bayes estimate for the parameter η of Exponential, Gamma, Log-Normal, and Weibull priors using equation [14] are given by:

$$\hat{\eta}_E = -\frac{1}{m}\log\left[\begin{array}{c} e^{-m\hat{\eta}} + \frac{1}{2}m^2e^{-m\hat{\eta}}\sigma_{\eta\eta} + me^{-m\hat{\eta}}(a_1)\sigma_{\eta\eta} + me^{-m\hat{\eta}}(a_2)\sigma_{\eta\delta} \\ -\frac{me^{-m\hat{\eta}}}{2}\sigma_{\eta\eta}S_1 - \frac{me^{-m\hat{\eta}}}{2}\sigma_{\delta\eta}S_2 \end{array}\right] \quad [32]$$

$$\hat{\eta}_G = -\frac{1}{m}\log\left[\begin{array}{c} e^{-m\hat{\eta}} + \frac{1}{2}m^2e^{-m\hat{\eta}}\sigma_{\eta\eta} - me^{-m\hat{\eta}}\left(\frac{a_3-1}{\hat{\eta}} - b_1\right)\sigma_{\eta\eta} \\ -me^{-m\hat{\eta}}\left(\frac{a_4-1}{\hat{\delta}} - b_2\right)\sigma_{\eta\delta} - \frac{me^{-m\hat{\eta}}}{2}\sigma_{\eta\eta}S_1 - \frac{me^{-m\hat{\eta}}}{2}\sigma_{\delta\eta}S_2 \end{array}\right] \quad [33]$$

$$\hat{\eta}_{LN} = -\frac{1}{m}\log\left[\begin{array}{c} e^{-m\hat{\eta}} + \frac{1}{2}m^2e^{-m\hat{\eta}}\sigma_{\eta\eta} - me^{-m\hat{\eta}}\left(-\frac{1}{\hat{\eta}} - \frac{log\hat{\eta} - a_5}{\hat{\eta}b_3^2}\right)\sigma_{\eta\eta} \\ -me^{-m\hat{\eta}}\left(-\frac{1}{\hat{\delta}} - \frac{log\hat{\delta} - a_6}{\hat{\delta}b_4^2}\right)\sigma_{\eta\delta} - \frac{me^{-m\hat{\eta}}}{2}\sigma_{\eta\eta}S_1 - \frac{me^{-m\hat{\eta}}}{2}\sigma_{\delta\eta}S_2 \end{array}\right] \quad [34]$$

$$\hat{\eta}_W = -\frac{1}{m}\log\left[\begin{array}{c} e^{-m\hat{\eta}} + \frac{1}{2}m^2e^{-m\hat{\eta}}\sigma_{\eta\eta} - me^{-m\hat{\eta}}\left(\frac{a_7-1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7-1}\right)\sigma_{\eta\eta} \\ -me^{-m\hat{\eta}}\left(\frac{a_8-1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8-1}\right)\sigma_{\eta\delta} - \frac{me^{-m\hat{\eta}}}{2}\sigma_{\eta\eta}S_1 - \frac{me^{-m\hat{\eta}}}{2}\sigma_{\delta\eta}S_2 \end{array}\right] \quad [35]$$

The Bayes estimate for the parameter δ of Exponential, Gamma, Log-Normal, and Weibull priors using equation [14] are given by:

$$\hat{\delta}_E = -\frac{1}{m}\log\begin{bmatrix} e^{-m\hat{\delta}} + \frac{1}{2}m^2 e^{-m\hat{\delta}}\sigma_{\delta\delta} + me^{-m\hat{\delta}}(a_1)\sigma_{\delta\eta} + me^{-m\hat{\delta}}(a_2)\sigma_{\delta\delta} \\ -\frac{me^{-m\hat{\delta}}}{2}\sigma_{\eta\delta}S_1 - \frac{me^{-m\hat{\delta}}}{2}\sigma_{\delta\delta}S_2 \end{bmatrix} \quad [36]$$

$$\hat{\delta}_G = -\frac{1}{m}\log\begin{bmatrix} e^{-m\hat{\delta}} + \frac{1}{2}m^2 e^{-m\hat{\delta}}\sigma_{\delta\delta} - me^{-m\hat{\delta}}\left(\frac{a_3-1}{\hat{\eta}} - b_1\right)\sigma_{\delta\eta} \\ -me^{-m\hat{\delta}}\left(\frac{a_4-1}{\hat{\delta}} - b_2\right)\sigma_{\delta\delta} - \frac{me^{-m\hat{\delta}}}{2}\sigma_{\eta\delta}S_1 - \frac{me^{-m\hat{\delta}}}{2}\sigma_{\delta\delta}S_2 \end{bmatrix} \quad [37]$$

$$\hat{\delta}_{LN} = -\frac{1}{m}\log\begin{bmatrix} e^{-m\hat{\delta}} + \frac{1}{2}m^2 e^{-m\hat{\delta}}\sigma_{\delta\delta} - me^{-m\hat{\delta}}\left(-\frac{1}{\hat{\eta}} - \frac{log\hat{\eta} - a_5}{\hat{\eta}b_3^2}\right)\sigma_{\delta\eta} \\ -me^{-m\hat{\delta}}\left(-\frac{1}{\hat{\delta}} - \frac{log\hat{\delta} - a_6}{\hat{\delta}b_4^2}\right)\sigma_{\delta\delta} - \frac{me^{-m\hat{\delta}}}{2}\sigma_{\eta\delta}S_1 - \frac{me^{-m\hat{\delta}}}{2}\sigma_{\delta\delta}S_2 \end{bmatrix} \quad [38]$$

$$\hat{\delta}_W = -\frac{1}{m}\log\begin{bmatrix} e^{-m\hat{\delta}} + \frac{1}{2}m^2 e^{-m\hat{\delta}}\sigma_{\delta\delta} - me^{-m\hat{\delta}}\left(\frac{a_7-1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7-1}\right)\sigma_{\delta\eta} \\ -me^{-m\hat{\delta}}\left(\frac{a_8-1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8-1}\right)\sigma_{\delta\delta} - \frac{me^{-m\hat{\delta}}}{2}\sigma_{\eta\delta}S_1 - \frac{me^{-m\hat{\delta}}}{2}\sigma_{\delta\delta}S_2 \end{bmatrix} \quad [39]$$

4.3 Lindley's Approximation of η and δ using ENTROPY:

The Bayes estimate for the parameter $\eta\eta$ of Exponential, Gamma, Log-Normal, and Weibull priors using equation [14] are given by:

$$\hat{\eta}_E = \Big[\hat{\eta}^{-c} + (\frac{\sigma_{\eta\eta}}{2}c(c+1)\hat{\eta}^{-c-2}\Big) + (c\hat{\eta}^{-c-1})(a_1)\sigma_{\eta\eta} + (c\hat{\eta}^{-c-1})(a_2)\sigma_{\eta\delta} - \frac{(c\hat{\eta}^{-c-1})}{2}\sigma_{\eta\eta}S_1 - \frac{(c\hat{\eta}^{-c-1})}{2}\sigma_{\delta\eta}S_2]^{-\frac{1}{c}} \quad [40]$$

$$\hat{\eta}_G = \Big[\hat{\eta}^{-c} + (\frac{\sigma_{\eta\eta}}{2}c(c+1)\hat{\eta}^{-c-2}\Big) - (c\hat{\eta}^{-c-1})\left(\frac{a_3-1}{\hat{\eta}} - b_1\right)\sigma_{\eta\eta} - (c\hat{\eta}^{-c-1})\left(\frac{a_4-1}{\hat{\delta}} - b_2\right)\sigma_{\eta\delta} - \frac{(c\hat{\eta}^{-c-1})}{2}\sigma_{\eta\eta}S_1 - \frac{(c\hat{\eta}^{-c-1})}{2}\sigma_{\delta\eta}S_2]^{-\frac{1}{c}} \quad [41]$$

$$\hat{\eta}_{LN} = \Big[\hat{\eta}^{-c} + (\frac{\sigma_{\eta\eta}}{2}c(c+1)\hat{\eta}^{-c-2}\Big) - (c\hat{\eta}^{-c-1})\left(-\frac{1}{\hat{\eta}} - \frac{log\hat{\eta} - a_5}{\hat{\eta}b_3^2}\right)\sigma_{\eta\eta} - (c\hat{\eta}^{-c-1})\left(-\frac{1}{\hat{\delta}} - \frac{log\hat{\delta} - a_6}{\hat{\delta}b_4^2}\right)\sigma_{\eta\delta} - \frac{(c\hat{\eta}^{-c-1})}{2}\sigma_{\eta\eta}S_1 - \frac{(c\hat{\eta}^{-c-1})}{2}\sigma_{\delta\eta}S_2]^{-\frac{1}{c}} \quad [42]$$

$$\hat{\eta}_W = \Big[\hat{\eta}^{-c} + (\frac{\sigma_{\eta\eta}}{2}c(c+1)\hat{\eta}^{-c-2}\Big) - (c\hat{\eta}^{-c-1})\left(\frac{a_7-1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7-1}\right)\sigma_{\eta\eta} - (c\hat{\eta}^{-c-1})\left(\frac{a_8-1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8-1}\right)\sigma_{\eta\delta} - \frac{(c\hat{\eta}^{-c-1})}{2}\sigma_{\eta\eta}S_1 - \frac{(c\hat{\eta}^{-c-1})}{2}\sigma_{\delta\eta}S_2]^{-\frac{1}{c}} \quad [43]$$

The Bayes estimate for the parameter δ of Exponential, Gamma, Log-Normal, and Weibull priors using equation [14] are given by:

$$\hat{\delta}_E = [\hat{\delta}^{-c} + (c\hat{\delta}^{-c-1})(a_1)\sigma_{\delta\eta} + \frac{1}{2}(c(c+1)\hat{\delta}^{-c-2}\sigma_{\delta\delta}) + (c\hat{\delta}^{-c-1})(a_2)\sigma_{\delta\delta} - \frac{(c\hat{\delta}^{-c-1})}{2}\sigma_{\eta\delta}S_1 - \frac{(c\hat{\eta}^{-c-1})}{2}\sigma_{\delta\delta}S_2]^{-\frac{1}{c}} \quad [44]$$

$$\hat{\delta}_G = [\hat{\delta}^{-c} - (c\hat{\delta}^{-c-1})\left(\frac{a_3-1}{\hat{\eta}} - b_1\right)\sigma_{\delta\eta} + \frac{1}{2}(c(c+1)\hat{\delta}^{-c-2}\sigma_{\delta\delta} - (c\hat{\delta}^{-c-1})\left(\frac{a_4-1}{\hat{\delta}} - b_2\right)\sigma_{\delta\delta} - \frac{(c\hat{\delta}^{-c-1})}{2}\sigma_{\eta\delta}S_1 - \frac{(c\hat{\eta}^{-c-1})}{2}\sigma_{\delta\delta}S_2]^{-\frac{1}{c}} \quad [45]$$

$$\hat{\delta}_{LN} = [\hat{\delta}^{-c} - (c\hat{\delta}^{-c-1})\left(-\frac{1}{\hat{\eta}} - \frac{log\hat{\eta} - a_5}{\hat{\eta}b_3^2}\right)\sigma_{\delta\eta} + \frac{1}{2}(c(c+1)\hat{\delta}^{-c-2}\sigma_{\delta\delta}) - (c\hat{\delta}^{-c-1})\left(-\frac{1}{\hat{\delta}} - \frac{log\hat{\delta} - a_6}{\hat{\delta}b_4^2}\right)\sigma_{\delta\delta} - \frac{(c\hat{\delta}^{-c-1})}{2}\sigma_{\eta\delta}S_1 - \frac{(c\hat{\eta}^{-c-1})}{2}\sigma_{\delta\delta}S_2]^{-\frac{1}{c}} \quad [46]$$

$$\hat{\delta}_W = [\hat{\delta}^{-c} - (c\hat{\delta}^{-c-1})\left(\frac{a_7-1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7-1}\right)\sigma_{\delta\eta} + \frac{1}{2}(c(c+1)\hat{\delta}^{-c-2}\sigma_{\delta\delta}) - (c\hat{\delta}^{-c-1})\left(\frac{a_8-1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8-1}\right)\sigma_{\delta\delta} - \frac{(c\hat{\delta}^{-c-1})}{2}\sigma_{\eta\delta}S_1 - \frac{(c\hat{\eta}^{-c-1})}{2}\sigma_{\delta\delta}S_2]^{-\frac{1}{c}} \quad [47]$$

5. SIMULATION STUDY

This study was conducted to compare the performance of Bayes estimates under different loss functions for the Topp-Leone Exponential distribution. Data sets of sizes n=20,50 and 100 representing small, moderate and large samples respectively, were generated with hyperparameters $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = m = 1$, $b_1 = b_2 = 1.5$, $c = 0.5$, $b_3 = b_4 = 1$, $b_5 = b_6 = 2$ and N=5000 replications. The simulation results for estimating the shape and scale parameters with different loss functions using identical priors are presented in Tables 3-4, utilizing the R package.

5.1 Results and Discussion

A comparative study based on Bayes risk for different loss functions to estimate the parameters of the Topp-Leone Exponential distribution is summarized in Table:3-4.

Bayes estimate of the parameters along with their Bayes Risk * for identical prior with different loss functions when $\eta = 0.5 \; and \; \delta = 1$
$\eta = 0.5 \; and \; \delta = 1$.

Table 3

SAMPLE SIZES	LOSS FUNCTIONS	PARAMETERS	PRIORS			
			EXPONENTIAL	GAMMA	LOG-NORMAL	WEIBULL
20	DEGROOT	η	0.60293 (0.066475)	0.568878 (0.06838)	0.669454 (0.039294)	0.632375 (0.059097)
		δ	1.28250 (0.1125307)	1.1368 (0.113649)	1.475939 (0.064673)	1.391362 (0.09713)
	LINEX	η	0.549001 (0.012573)	0.518817 (0.010519)	0.629877 (0.011411)	0.580594 (0.013218)
		δ	1.064835 (0.074388)	0.976691 (0.045472)	1.283620 (0.090761)	1.164008 (0.092274)
	ENTROPY	η	0.532706 (0.008395)	0.505298 (0.007186)	0.637953 (0.007949)	0.562756 (0.008395)
		δ	1.048895 (0.013683)	0.961867 (0.010017)	1.270560 (0.014653)	1.149341 (0.015549)
50	DEGROOT	η	0.539982 (0.027139)	0.530182 (0.027842)	0.565995 (0.021759)	0.5494 (0.027139)
		δ	1.114700 (0.049272)	1.078422 (0.050909)	1.187738 (0.038550)	1.148499 (0.045834)
	LINEX `	η	0.521148 (0.004037)	0.51136 (0.003932)	0.54978 (0.003728)	0.53105 (0.00403)
		δ	1.031183 (0.029049)	0.997515 (0.026656)	1.112649 (0.029183)	1.0662 (0.030094)
	ENTROPY	η	0.514218 (0.003463)	0.504766 (0.003331)	0.553462 (0.003343)	0.523951 (0.003463)
		δ	1.021257 (0.006197)	0.988519 (0.005705)	1.101983 (0.006283)	1.055699 (0.006429)
100	DEGROOT	η	0.518808 (0.013712)	0.51428 (0.013903)	0.531918 (0.012256)	0.523255 (0.013712)
		δ	1.056347 (0.025385)	1.039676 (0.025865)	1.093482 (0.022376)	1.072476 (0.024445)
	LINEX	η	0.509804 (0.001861)	0.505258 (0.001844)	0.523584 (0.001781)	0.514373 (0.001856)
		δ	1.015836 (0.013843)	0.99957 (0.013384)	1.0553295 (0.0137501)	1.032392 (0.014013)
	ENTROPY	η	0.506328 (0.001736)	0.501856 (0.001706)	0.5252350 (0.0017025)	0.510861 (0.001736)
		δ	1.010079 (0.003191)	0.994096 (0.003075)	1.0493324 (0.0032090)	1.026456 (0.003245)

* Bayes Risk are given in the parenthesis.

Bayes estimate of the parameters along with their Bayes Risk * for identical prior with different loss functions when $\eta = 1 \; and \; \delta = 1$
$\eta = 1 \; and \; \delta = 1$.

Table 4

SAMPLE SIZES	LOSS FUNCTIONS	PARAMETERS	PRIORS			
			EXPONENTIAL	GAMMA	LOG-NORMAL	WEIBULL
20	DEGROOT	η	1.19557 (0.087745)	0.587241 (0.064522)	1.380657 (0.059055)	1.318152 (0.087745)
		δ	1.16295 (0.077983)	1.044462 (0.067412)	1.316177 (0.054472)	1.258488 (0.069892)
	LINEX	η	1.04219 (0.050776)	0.954938 (0.01798)	1.218846 (0.072767)	1.14139 (0.071614)
		δ	1.03005 (0.044047)	0.950922 (0.02583)	1.190186 (0.052897)	1.117588 (0.053853)
	ENTROPY	η	1.030779 (0.009151)	0.944224 (0.005293)	1.001292 (0.010817)	1.132481 (0.009151)
		δ	1.017738 (0.008829)	0.941671 (0.005936)	1.176555 (0.009957)	1.104033 (0.010233)
50	DEGROOT	η	1.083547 (0.034225)	1.048591 (0.034513)	1.142298 (0.028866)	1.116139 (0.034225)
		δ	1.068899 (0.033231)	1.036301 (0.033439)	1.125085 (0.028216)	1.09951 (0.031374)
	LINEX	η	1.025803 (0.020069)	0.994102 (0.017927)	1.088047 (0.020593)	1.058753 (0.020962)
		δ	1.01566 (0.018054)	0.985772 (0.016337)	1.075033 (0.018413)	1.046562 (0.018755)
	ENTROPY	η	1.019124 (0.004167)	0.988146 (0.00375)	1.053184 (0.004291)	1.051676 (0.004167)
		δ	1.008683 (0.004043)	0.979589 (0.003656)	1.06748 (0.004172)	1.03912 (0.004218)
100	DEGROOT	η	1.039334 (0.01709)	1.023898 (0.017228)	1.067826 (0.015651)	1.054301 (0.01709)
		δ	1.033935 (0.016888)	1.018909 (0.016992)	1.061929 (0.015524)	1.048523 (0.016385)
	LINEX	η	1.012196 (0.009255)	0.99727 (0.00888)	1.04171 (0.00927)	1.02736 (0.009382)
		δ	1.007745 (0.008823)	0.993234 (0.008469)	1.036651 (0.008855)	1.022482 (0.008951)
	ENTROPY	η	1.008337 (0.002117)	0.993628 (0.002025)	1.03264 (0.002141)	1.023383 (0.002117)
		δ	1.003854 (0.002086)	0.989561 (0.001995)	1.032598 (0.002116)	1.018462 (0.002127)

* Bayes Risks are given in the parenthesis.

The following results concerning different loss functions with identical priors are observed as follows:

(i)for the scale parameter $\delta = 1$ and shape parameter $\eta = 0.5$

- **DEGROOT Loss Function**: The Log-Normal prior exhibits a lower Bayes risk for shape and scale parameters as sample size increases.
- **LINEX Loss Function**: Gamma prior shows lower Bayes risk for both parameters at a sample size of 20. At sample sizes 50 and 100, Log-Normal prior is preferred for the shape parameter, while Gamma prior is preferred for the scale parameter.
- **ENTROPY Loss Function**: Gamma prior shows lower Bayes risk for both parameters at sample sizes 20 and 50. At sample size 100, Log-Normal prior is preferred for the shape parameter, while Gamma prior is preferred for the scale parameter.

(ii)for the scale parameter $\delta = 1$ and shape parameter $\eta = 1$

- **DEGROOT Loss Function**: Log-Normal prior shows lower Bayes risk for shape and scale parameters.
- **LINEX and ENTROPYLoss Functions:** Gamma prior shows lower Bayes risk for shape and scale parameters

Comparing DEGROOT, LINEX, and ENTROPY loss functions across different priors, the combination of Log-Normal prior for the shape parameter and Gamma prior for the scale parameter shows consistently lower Bayes risk with the ENTROPY loss function.

5.2 Real Data Set:

This data set is taken from M.D. Nicholas. et al (2006) and the data represent the tensile strength of 100 observations of carbon fibres. The results are presented in Table 5.

Bayes estimate of the parameters along with their Bayes Risk *
for identical prior with different loss functions

Table 5

LOSS FUNCTIONS	PARAMETERS	PRIORS			
		EXPONENTIAL	GAMMA	LOG-NORMAL	WEIBULL
DEGROOT	η	2.54771 (0.019914)	2.472686 (0.018527)	2.646805 (0.019188)	2.618629 (0.019914)
	δ	0.31861 (0.008936)	0.314108 (0.008603)	0.323512 (0.008854)	0.322994 (0.008885)
LINEX	η	2.439961 (0.05701)	2.380377 (0.046496)	2.530708 (0.06531)	2.50332 (0.063746)
	δ	0.315318 (0.000447)	0.310989 (0.000417)	0.320188 (0.000459)	0.319665 (0.000459)
ENTROPY	η	2.462061 (0.002258)	2.39772 (0.001874)	1.467531 (0.002496)	2.529028 (0.002258)
	δ	0.313713 (0.001068)	0.309528 (0.000976)	0.318503 (0.001117)	0.317984 (0.001114)

* Bayes Risks are given in the parenthesis.

From the above Table 5, we observe that Gamma prior has a lower Bayes risk. In the case of shape and scale parameters, the ENTROPY and LINEX loss functions perform better.

6. CONCLUSION

In this study, we have discussed the problem of Bayesian estimation for the Topp-Leone Exponential distribution with identical priors under an asymmetric loss function by applying Lindley's approximation method and illustrated the methodology through simulation technique and real data set. On comparing the estimated Bayes risk values of the Topp-Leone Exponential distribution using asymmetric loss functions, it is found that the risk under the Entropy loss function is the minimum among the other loss functions. Finally, it is observed that two parameters Topp-Leone Exponential distribution with gamma prior under Entropy loss function performed well in this study. Further studies are necessary to confirm these findings.

REFERENCES

1. Al-Shomrani A, Arif O, Ibrahim S, Hanif S and Shahbaz M (2016), "Topp–Leone Family of Distributions: Some Properties and Application", Pakistan Journal of Statistics and Operation Research, 12(3), 443-451.
2. Anitta SA, and Dais George (2020), "Bayesian Analysis of Two Parameter Weibull Distribution using Different Loss Functions", Stochastic Modeling and Applications,24(2).

3. Epstein B, and Sobel M (1954), “Some theorems to life resting form an exponential distribution”, Annals of Mathematical Statistics, 25(2):373-381.
4. Farouk Metiri, Halim Zeghdoudi, and Mohamed Riad Remiata, On Bayes estimates of Lindley distribution under Linex loss function: Informative and Non-informative priors, Global Journal of Pure and Applied Mathematics 12 (2016), 391- 400.
5. Fatoki Olayode (2019), “The Topp-Leone Rayleigh Distribution with Application”,American Journal of Mathematics and Statistics, 9 (6): 215-220.
6. Fithriani I, Arief Hakim, and Mila Novita (2019), “A comparison of the Bayesian method under symmetric and asymmetric loss functions to estimate the shape parameter K of Burr distribution”, Journal of Physics Conference Series (2019),1218.
7. Genc A (2012), “ Moments of order statistics of Topp Leone distribution”, Statistical Papers,53(1):117-31.
8. Hind Jawad Kadhim Albderia (2021), “Estimate survival function of the Topp-Leone exponential distribution with application”, International journal Nonlinear Analysis and Applications,12(2): 53-60.
9. Kawsar F, Ahmed SP (2017), “Bayesian Approach in Estimation of shape parameter of Exponentiated moment Exponential distribution” Journal of Statistical Theory and Applications,17(2):359-74.
10. Kotz S, Seier E (2007), “ Kurtosis of the Topp Leone distributions”, Interstat, 1-15.
11. Lindley DV, (1980),Approximate Bayesian methods, Journal of Statistical Computation and Simulation. Trabajos de Estadistica y de Investigacion Operativa 31,223-45.
12. Mohammed H, AbuJarad, Athar Ali Khan (2018), “Bayesian Survival Analysis of Topp-Leone Generalized Family with Stan”, International Journal of Statistics and Applications,8(5):274-90.
13. Noman Rasheed (2019), “Topp–Leone compound Rayleigh distribution: properties and applications”, Research Journal of Mathematical and Statistical Science,7(3): 51–58.
14. Nadarajah S, Kotz S (2003), “Moments of some J-shaped distributions”, Journal of Applied Statistics, 30(3):311-17.
15. Nicholas MD and Padgett WJ (2006), “A Bootstrap control chart for Weibull percentiles,”,Quality and Reliability engineering international ,22 : 141-151.
16. Radha RK, Venkatesan P (2013), “Bayes Estimator as a function of some classical estimators for Power Function Distribution”, International Journal of Statistics and Analysis,3(2):105-09.
17. Randhir Singh (2021), “Bayesian estimation of the unknown parameter and reliability of the Exponential distribution with a non-natural conjugate prior”,Journal of Emerging Technologies and Innovative Research, 8: 228-236.
18. Singh SK, Umesh Singh, and Dinesh Kumar (2011), “Estimation of parameters and reliability function of Exponentiated Exponential distribution: Bayesian approach under General Entropy loss function”, Pakistan Journal of Statistics and Operation Research, 7:199-216.
19. Saridha D, Radha RK and Venkatesan P(2024), “Bayesian estimation of Topp-Leone Exponential distribution using symmetric loss functions for identical priors”, Sirjana Journal, 54(3), 19-26.
20. Topp CW and Leone FC.(1955), “A family of J-shaped frequency functions”, Journal of the American Statistical Association, 50: 209-19.