Alpha Distribution and the Economic Resilience System

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ABSTRACT

A constant concern in the field of mechanical engineering is finding the balance between the costs of machining components and the costs of replacing worn processing devices. Determining the optimal moment, to replace processing tools, involves describing their resilience, due to the stress in use, through an appropriate statistical model, which statistically reproduces the behavior under extreme demands. One such model is the Alpha distribution proposed by Drujinin in 1967. The first applications were in engineering practice, in the field of metal processing, the model best describing the behavior of the processing device to the resistance of the processed metal, the stress exerted on the cutting tool, as well as its subsequent resilience.

Along with the bibliographic investigation and the presentation of the distribution's genesis from a historical point of view, this article presents the estimation of the model's main parameters, the establishment of the lower tolerance limit and a practical application. Somehow forgotten in the specialized literature, the Alpha distribution has numerous applications, exceeding the boundaries of mechanical engineering, such as the modeling of effects, but especially macroeconomic resilience processes following crises, such as the 2008 global slump or the one generated by the COVID-19 pandemic.

Keywords: *Alpha distribution, economic resilience, parameter estimation, extreme loads, cutting tools, tolerance limits*

JEL Classification: *C13, C21, C46, L15, L62, O14*

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1. INTRODUCTION

Entire industries, a wide range of products from household products to automobiles, airplanes, drilling rigs, sea and river ships, depend on the performance of metalworking devices. The knives of the cutting processing devices are simultaneously subjected to technological loads and to mechanical, chemical and thermal attacks. The interest is major to model such processes and to estimate, as accurately as possible, the service life of these devices, due to the high costs generated by industrial users in the field of metal processing, the costs are ultimately found in the price of the products sold.

The Alpha distribution, due to its capacity to quantitatively reflect some processes, can help to improve the design of some systems under the conditions of optimizing the costs between manufacturing and operation. Obviously its scope of application goes beyond the field of metal processing, where this distribution was launched, and it is recommended wherever some processes are to be modelled in which extreme stress can cause failures, such as: the landing gear of aircrafts, the load for aircraft engines to take off, modelling the loads of sloping traction systems on locomotives, modelling the behaviours of the components of propulsion systems on spacecraft, and even the reliability of the human body to extreme demands, such as athletes, fighter pilots, cosmonauts etc.

Such a model was proposed in Russia by Drujinin [1] developed by Katzev [2] used and then completed in Romania by Dorin and Vodă [3], respectively Vodă [4], which develops the issue of sustainability modelling. The model was then taken over by Koutras and Tsokos [5, 6] and resumed by Savchuk and Tsokos [7]. Overall, from the perspective of the time elapsed since the appearance of this distribution (aspect that will be developed in the part intended to investigate the literature) the impact was a minor one in the international literature, most likely both due to the language barrier, as Russian and Romanian have a limited circulation, and to the circulation limits of scientific information in the 1970s, and then the subject was somewhat abandoned, although the model provides a number of facilities in modelling processes in which extreme loads occur, due to its remarkable versatility to adapt to specific processes of a great variety.

Thus, the distribution has applicability in the modeling of processes in which extreme loads occur, of the technical systems' resilience, but also of economic-social macrosystems following crises.

2. LITERATURE REVIEW

Over time, a rich literature has been established on the reliability of cutting tools, contributions that aim to highlight the diverse range of methods proposed for estimating their reliability, and reaching the Alpha distribution has been the result of gradual accumulations. Thus, Sherif, 15 years away from Drujinin, develops a version of the truncated normal reverse distribution, as a model of reliability for systems that suffer a high rate of wear and failure, but with a different truncation point compared to the Drujinin model. Sherif [8] argues that tool defects in cutting processes may be due to sudden stress, environmental aggression, hidden defects, violation of procedures in the manufacturing process or improper use. Moreover, he considers this distribution to be appropriate for the case of high-speed cutting tools, so he also cites the work of Katzev [2], but under the name of Katsey. The field of reliability and durability of cutting tools has proved fertile in statistical modelling, including using the Alpha model. Both theoretical and experimental research has been a real development of descriptive models of cutting processes. Dorin and Vodă [3], then Vodă [4], effectively used the Alpha model in data processing, coming from reliability tests on some drills, and developed maximum likelihood estimators for the parameters of the repair proposed by Drujinin [1]. Koutras and Tsokos [5, 6] resume and develop the subject treated by Dorin and Vodă [3] and present it in the journal *Statistica* in Italy. An original element is given by the quotation of Katzev which appears erroneously in the bibliography V.M. Katzev instead of P.G. Katzev. The article in *Statistica* (Bologna) is later included in the volume "Bayesian Theory and Methods with Applications", coordinated by Savchuk and Tsokos [7], a volume that brings together new practical applications in the fields of health sciences, engineering, environmental sciences, business and economics, as well as the social sciences.

In the last edition of the famous treatise on statistical distributions [9] Johnson & al also approaches the Alpha distribution very synthetically (p 173), Salvia [10] being invoked as the initiator of the model, and he makes a reference on the Alpha distribution (he also cites Katzev as Katsey in the paper). Vladimirescu and Tunaru [11] proposed hypothesis tests for discrimination between the populations of two Alpha distributions. The tests developed here are uniformly most powerful unbiased and can be used to test various general hypotheses related to this probability distribution which is less known by professional statisticians. Wager and Barash [12], on the basis of numerous tests performed on high-speed steel tools used in the processing of low-carbon materials, obtained results indicating that the service life of these tools corresponds to statistical distributions that can be approximated by the normal distribution, with a coefficient of variation of about 0.3. Wiklund [13] prefers a Bayesian approach, from an engineering perspective, to describe the cutting processing and makes a prediction of the service life of cutting tools. Gertsbakh [14] develops a university treatise, which focuses primarily on the maintenance, collection and processing of statistical data in order to optimize preventive interventions. Rausand and Høyland [15] carry out a complete and updated theory of the reliability of mechanical processing systems, this work being at its second edition, which is enriched with the latest acquisitions and developments. Pearn and Hsu [16] propose a model for replacing cutting tools by establishing an alert level based on the defective fraction. Lin [17] provides a study in which the failure rate is used to describe the reliability of cutting tools, by constructing a reliability equation, based on the risk function dependent on the reliability of the cutting tool. Deng & al [18] propose a tool replacement model in the processing process based on the log-normal distribution, aiming to achieve a balance between the costs of tool replacement and the cost of damage caused by quality defects of products resulting from the use of worn out processing devices.

Karandikar [19] assumes that the tool life decreases with increasing cutting speed, to which additional factors can be added, such as the nature of the material being processed, the service life of the processing tools being generally considered a stochastic process. As a method, he uses Bayesian inference to make a model for estimating tool life. Salonitis and Kolios [20] use stochastic methods and modelling, by using low-volume data sets in wear and tear modelling, including the simulation by Monte Carlo method. Hsu and Shu [21] propose to evaluate the reliability and determine the optimal replacement time of a machine tool under conditions of tool damage, using an inhomogeneous Markov process combined with a cost function. Nadarajah and Kotz [22], in the line of generalizing some classical distributions, proceed to develop one of the most frequently used distributions in reliability modelling, namely the exponential distribution, building the variant called by the authors "exponential beta", based on the logit of a random beta variable, and based on the generating function, the calculation of the first four moments, as well as of the entropic indicators is performed. The authors propose the introduction of an alternative form of generating the asymmetric normal distribution, which allows the framing of unimodal and bimodal data sets. The basic properties of this new distribution, the moments, the maximum probability and the Fisher information matrix are studied, and the theoretical development is strengthened with a real-life application.

Vagnorius & al [23] aim to optimize tool replacement times, balancing the replacement moments with the costs generated by premature replacement, respectively the losses generated by the deterioration of product quality, and they propose in the wear and tear modelling process a combination of Weibull distributions and Poisson processes. Wang & al. [24] provide a study based on a mathematical model of dynamic reliability for the machining process, using the failure rate, finalized by proposing an algorithm to identify the optimal time to replace the processing tool. Chen & al [25] consider that traditional methods of assessing reliability, based on large samples, are inefficient for numerically controlled cutting tools and machining robots, and propose the semi-normal function as a solution to define the operational reliability of the cutting tool. Kalpakjian and Schmid [26] provide detailed descriptions of modern manufacturing processes and operations in a state-of-the-art manufacturing and technology manual, including QR codes that provide access to video examples. Gaddafee and Chinchanikar [27] are the editors of a volume that brings together the selection of papers presented at a symposium on the wear and tear of metalworking tools and the modelling of these processes. Elal-Olivero [28] introduces an alternative to generate a normal distribution with strong asymmetry. The basic properties of this distribution are studied, such as stochastic representation, moments, maximum probability and informational matrix. Shafiei & al [29] introduce a new class of asymmetric distributions, by extending the normal alpha skew distribution proposed by Elal Olivero, developing the statistical properties of this new distribution family. The moments and shape parameters are developed, including the slope, the kurtosis coefficients and the generation function, being appreciated the quality of the estimators obtained through a Monte Carlo simulation study. Cordeiro and de Castro [30] develops generalized forms of some classical distributions, being identified by the authors by adding the prefix “Kw” (as this distribution was originally proposed by Poondi Kumaraswamy) for example the Kw-normal distribution, Kw-Weibull, Kw-gamma, Kw-Gumbel and Kw-inverse Gaussian etc. Kumaraswamy’s double bounded distribution [31] is a family of continuous probability distributions defined on the interval (0,1). It is similar to the Beta distribution, but much simpler to use especially in simulation studies since its probability density function, cumulative distribution function and quantile functions can be expressed in closed form.

In 1978 James J. Filliben released Dataplot, a free public-domain, multi-platform software system for scientific visualization, statistical analysis, and non-linear modeling [32]. This allow to compute the alpha probability density function (ALPPDF) and also related commands that: compute the alpha cumulative distribution function (ALPCDF), compute the alpha percent point function (ALPPPF), compute the alpha hazard function (ALPHAZ), compute the alpha cumulative hazard function (ALPCHAZ). The sources of the statistical method used are Johnson et al [9] (p. 173) and Salvia [10] (pp. 251-252). Likewise, SciPy.org [33], a Python-based ecosystem of open-source software for mathematics, science, and engineering, provides functions for Alpha distribution, using the same sources [9] and [10] for the statistical methods.

3. DESCRIPTION OF THE ALPHA MODEL

The distributed random variable Alpha represents the inverse of the normal variable $N(t;\mu,\sigma^2)$ truncated at the origin.

Indeed, if X is a random variable with the density $f_x(x;\theta)$, $x \in R$, $\theta > 0$ and x_t is a truncation point to the left, i.e. $x > x_t$, then the density of x_t is

$$x_t\colon f_{x_t}(x;\theta) = \frac{1}{1-F_x(x_t;\theta)} \cdot f_x(x;\theta), x > x_t \tag{1}$$

Here $F_x(x_t;\theta)$ is the distribution function of X, calculated at the truncation point. If $F_x(x;\theta)$ is the normal distribution function, then the normal density truncated to the left has the following form:

$$x_t\colon f_{x_t}(x;\mu,\sigma^2) = \frac{(\sigma\sqrt{2\pi})^{-1}}{1-F_0\left(\frac{x_t-\mu}{\sigma}\right)} \cdot exp\left[-\frac{1}{2}\left(\frac{x_t-\mu}{\sigma}\right)^2\right], x > x_t \tag{2}$$

where:

$$F_0(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} exp(-t^2/2)dt \tag{3}$$

It is clear that if the truncation point is $x_t = 0$, then:

$$f_0(x;\mu,\sigma^2) = \frac{1}{(\sigma\sqrt{2\pi})F_0\left(\frac{\mu}{\sigma}\right)} \exp\left[-\frac{1}{2}\left(\frac{x_t-\mu}{\sigma}\right)^2\right], x > 0 \tag{4}$$

As the distribution function of the variable X_t^{-1} is:

$$F(y) = Prob\{X_t^{-1} < y\} = 1 - Prob\left\{X_t < \frac{1}{y}\right\} =$$
$$= 1 - \left(\sigma\sqrt{2\pi}\right)^{-1}\left[F_0\left(\frac{\mu}{\sigma}\right)\right]^{-1}\int_0^{\frac{1}{y}} \exp\left[-\frac{1}{2}\left(\frac{x_t - \mu}{\sigma}\right)^2\right]dx \quad (5)$$

The following form results in the end:

$$F^{`}(y) = \frac{\left(\sigma\sqrt{2\pi}\right)^{-1}}{y^2 F_0\left(\frac{\mu}{\sigma}\right)} \cdot exp\left[-\frac{1}{2}\left(\frac{\frac{1}{y}-\mu}{\sigma}\right)^2\right], y > 0 \quad (6)$$

The name *Alpha distribution* assigned by Drujinin is given by the notation $\mu = \frac{\alpha}{\beta}$, $\sigma = 1/\beta$, $\alpha, \beta > 0$ so that the density of the function resulted in the form:

$$T: f_T(t; \alpha, \beta) = \frac{\beta}{t^2 F_0(\alpha)\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{\beta}{t} - \alpha\right)^2\right], x > 0;\ \alpha, \beta > 0 \quad (7)$$

(expression in which the notation *x* has been replaced by *t* to maintain the unity of the symbols in the following).

In relation to the distribution density (7) "α" is the essential parameter, a situation that made Drujinin to assign the name *Alpha* to the distribution. Katzev also remarked that, in most experimental cases, $\alpha \gg 2$. For example, if α is even equal to 2, then 0.98 results, and an increase by only 0.3 generates a value of 0.99 and $F_0\,(\alpha = 2.5) \cong 0.99959$ and, therefore, we can practically consider $F_0\,(\alpha > 2) \approx 1$. In which case the density in the form (8) may be used without any significant loss of precision:

$$\tilde{f}_T(t; \alpha, \beta) = \frac{\beta}{t^2\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{\beta}{t} - \alpha\right)^2\right], t > 0;\ \alpha, \beta > 0 \quad (8)$$

Tsokos and Koutras [6,] took over and analysed the distribution under this form.

The main elements that describe the Alpha distribution:

- Distribution function: $\tilde{F}_T(t; \alpha, \beta) = 1 - F_0\left(\frac{\beta}{t} - \alpha\right)$ (9)
- Reliability function: $\tilde{R}_T(t; \alpha, \beta) = F_0\left(\frac{\beta}{t} - \alpha\right)$ (10)

• The mean value: $E(T) = \frac{\beta}{\alpha}\left(1 + \frac{1}{\alpha^2}\right)$ (11)

• The variance: $Var(T) = \frac{\beta^2}{\alpha^4}\left(1 + \frac{8}{\alpha^2}\right)$ (12)

• *The second quartile* (or the median): $t_{me} = \frac{\beta}{\alpha}$ (13)

(a note is required: if α is high enough, then the value of the mean coincides with the value of the median).

• The mode (or modal value): $t_{mo} = \frac{\beta}{4}\left(\sqrt{\alpha^2 + 8} - \alpha\right)$ (14)

• The *coefficient of variation:* $CV(T) = \frac{(\alpha^2+8)^{1/2}}{\alpha^2+1}$ (15)

• Disturbance coefficient: $SNR = \frac{(\alpha^2+1)}{(\alpha^2+8)^{1/2}}$ (16)

Signal-to-Noise Ratio (SNR) is an indicator borrowed from the engineering field, where it is used to compare the level of an expected signal with the background noise level. Informally taken over and used, the signal-to-noise ratio is sometimes used to refer to the relationship between useful information and irrelevant data, off-topic posts and spam are considered "noise" that interferes with the "signal" of relevant information. The signal-to-noise ratio is defined as the ratio of the strength of a signal to the strength of the background noise. In the processing of data from tests/ exploitation it has the significance of an indicator of disturbances.

4. ESTIMATION OF ALPHA DISTRIBUTION PARAMETERS

Based on the expressions of the indicators presented above, the method of least squares, the method of moments or method of maximum likelihood estimation, can be used successfully to estimate the parameters α and β.

The method of moments is fast enough and ensures sufficient accuracy in practical applications. Starting from the coefficient of variation, that does not depend on the value of the parameter β, we can write:

$$\widehat{CV(T)} = \frac{s}{\bar{t}} \tag{17}$$

where $\bar{t}$ and s are the mean, respectively the standard deviation, established on the basis of the sampling data.

The value of β results easily if the theoretical mean (11) is equaled with the survey mean, obtained from the experimental data, therefore:

$$\bar{t} = \frac{\hat{\beta}}{\hat{\alpha}}\left(1 + \frac{1}{\alpha^2}\right) \tag{18}$$

The equation that generates the β estimator:

$$\hat{\beta} = \frac{\hat{\alpha}^2 \bar{t}}{1+\hat{\alpha}^2} \tag{19}$$

The estimations of parameters, using the same method of moments, can also be completed using the median. Thus, if t_{me} is the median of the sample:

$$\widehat{t_{me}} = \begin{cases} t_{(k+1)}, if\ n = 2k+1 \\ \frac{1}{2}\left[t_{(k)} + t_{(k+1)}\right], if\ n = 2k \end{cases} \tag{20}$$

Therefore: $\hat{t}_{me} = \frac{\hat{\beta}}{\hat{\alpha}}$

and by equaling the theoretical median (13) we get:

$$\hat{\beta} = \hat{\alpha} \cdot t_{(k+1)} \tag{21}$$

hence:

$$2\hat{\beta}(\sqrt{\hat{\alpha}^2 + 8} - \hat{\alpha}) \tag{22}$$

and combined with $E(t) = \bar{t}$ follow:

$$\hat{\alpha} = \left[\frac{t_{(k+1)}}{\bar{t} - t_{(k+1)}}\right]^{1/2} \tag{23}$$

Based on the modal value (14):

$$t_{mo} = \frac{\beta}{4}\left(\sqrt{\alpha^2 + 8} - \alpha\right)$$

the following solutions are obtained by a simple calculation:

$$\hat{\alpha} = \sqrt{6} \cdot t_{mo}/3t_{me} \qquad (24)$$

$$\hat{\beta} = 2\sqrt{6}t_{mo}/3 \qquad (25)$$

If the ratio between the modal value (14) and the median value (13) is used:

$$\frac{t_{mo}}{t_{me}} = \frac{\alpha}{4}\left(\sqrt{\alpha^2 + 8} - \alpha\right) < 1 \qquad (26)$$

Thus $t_{mo} < t_{me} < E(T)$ which means th at the distribution is asymmetric to the right. The values of the constant

$$K(\alpha) = \frac{\alpha}{4}\left(\sqrt{\alpha^2 + 8} - \alpha\right)$$

calculated (table 1) for
$\alpha = 2(0.1), 4(0.2), 5(0.25), 7(0.3), 8(0.4), 10(0.45), 15(0.5), 20(0.75), 30(1.0), 40, 50$
simplify, for practitioners, the obtaining of the α parameter value.

Values for $K(\alpha)$

Table 1

alpha	K	alpha	K	alpha	K	alpha	K	alpha	K
2.00	0.7321	3.80	0.8902	7.90	0.9699	15.00	0.9913	26.00	0.9971
2.10	0.7470	3.90	0.8947	8.00	0.9706	15.50	0.9918	26.75	0.9972
2.20	0.7608	4.00	0.8990	8.40	0.9732	16.00	0.9923	27.50	0.9974
2.30	0.7737	4.20	0.9068	8.80	0.9754	16.50	0.9928	28.25	0.9975
2.40	0.7857	4.40	0.9137	9.20	0.9774	17.00	0.9932	29.00	0.9976
2.50	0.7968	4.60	0.9200	9.60	0.9792	17.50	0.9936	29.75	0.9978
2.60	0.8072	4.80	0.9256	10.00	0.9808	18.00	0.9939	30.00	0.9978
2.70	0.8169	5.00	0.9307	10.45	0.9823	18.50	0.9942	31.00	0.9979
2.80	0.8260	5.25	0.9364	10.90	0.9837	19.00	0.9945	32.00	0.9981
2.90	0.8344	5.50	0.9414	11.35	0.9849	19.50	0.9948	33.00	0.9982
3.00	0.8423	5.75	0.9459	11.80	0.9860	20.00	0.9950	34.00	0.9983
3.10	0.8497	6.00	0.9499	12.25	0.9870	20.75	0.9954	35.00	0.9984
3.20	0.8567	6.25	0.9535	12.70	0.9879	21.50	0.9957	36.00	0.9985
3.30	0.8632	6.50	0.9567	13.15	0.9887	22.25	0.9960	37.00	0.9985
3.40	0.8693	6.75	0.9596	13.60	0.9894	23.00	0.9962	38.00	0.9986
3.50	0.8750	7.00	0.9622	14.05	0.9901	23.75	0.9965	39.00	0.9987
3.60	0.8804	7.30	0.9650	14.50	0.9907	24.50	0.9967	40.00	0.9988
3.70	0.8855	7.60	0.9676	14.95	0.9912	25.25	0.9969	50.00	0.9992

[1] Authors' own calculation

The estimation of the parameters by the method of moments can also be done using the SNR perturbation coefficient, using the values in the table with $SNR = (\alpha^2 + 1)/(\alpha^2 + 8)^{1/2}$ for different values of the α parameters. From the tables [34] and based on the value of the perturbation coefficient, the value of the α parameter is easily obtained.

An interesting property is that of the distribution of the first order statistics in an Alpha population:

$$F_{T_{(1)}}(t, \alpha, \beta) = 1 - [1 - F_T(t; \alpha, \beta)]^n = 1 - F_0^n\left(\frac{\beta}{t} - \alpha\right) \tag{27}$$

It is easy to show that $1 - F_{T_{(1)}}(t, \alpha, \beta)$ is precisely the reliability of an automatic processing line, in which the machines are arranged in series. Indeed, the Alpha reliability function being:

$$R_T(t; \alpha, \beta) = F_0\left(\frac{\beta}{t} - \alpha\right)$$

for a series type system its reliability will result from the shape R_S:

$$R_s = \prod_1^n R_T(t; \alpha_i \beta_i) = \prod_1^n F_0\left(\frac{\beta_i}{t} - \alpha_i\right)$$

If $\alpha_i = \alpha \alpha_i = \alpha$ and $\beta_i = \beta$ for any $i = \overline{1, n}$ then:

$$R_s = F_0^n\left(\frac{\beta}{t} - \alpha\right) R_s = F_0^n\left(\frac{\beta}{t} - \alpha\right) \tag{28}$$

which is exactly:

$$1 - F_{T_{(1)}}(t; \alpha, \beta) \tag{29}$$

In the case of parallel arrangement, the reliability is given by the expression:

$$R_p = 1 - \left[1 - F_0\left(\frac{\beta}{t} - \alpha\right)\right]^n = 1 - \left[1 - C_n^1 F_0\left(\frac{\beta}{t} - \alpha\right) + \right] + C_n^2 F_0^2\left(\frac{\beta}{t} - \alpha\right) - \cdots - (-1)^{n-1} C_n^n F_0^2\left(\frac{\beta}{t} - \alpha\right) = C_n^1 F_0\left(\frac{\beta}{t} - \alpha\right) - C_n^2 F_0^2\left(\frac{\beta}{t} - \alpha\right) + \cdots + (-1)^{n-1} C_n^n F_0^2\left(\frac{\beta}{t} - \alpha\right) = \sum_{k=1}^{n} (-1)^{k+1} C_n^k F_0^k\left(\frac{\beta}{t} - \alpha\right) \quad (30)$$

With regard to the estimation of the parameters using the maximum likelihood method, this generates results both if it is assumed to be known and if both parameters are unknown. The solutions developed in Dorin and Vodă [3], or Dorin & al [34] present the case when both parameters are unknown, a situation in which the system of maximum likelihood is the following:

$$\frac{\partial \ln L}{\partial \hat{\beta}} = \frac{n}{\hat{\beta}} - \hat{\beta} \sum_{i=1}^{n} \frac{1}{t_i} + \hat{\alpha} \sum_{i=1}^{n} \frac{1}{t_i} = 0 \quad (31)$$

$$\frac{\partial \ln L}{\partial \hat{\alpha}} = \hat{\beta} \sum_{i=1}^{n} \frac{1}{t_i} - n\hat{\alpha} = 0$$

a system that ultimately leads to the solutions:

$$\hat{\alpha} = \left[n \sum_1^n \frac{1}{t_i^2} - \left(\sum_1^n \frac{1}{t_i}\right)^2\right]^{-\frac{1}{2}} \cdot \sum_1^n \frac{1}{t_i} \text{ and}$$

$$\hat{\beta} = n\left[n \sum_1^n \frac{1}{t_i} - \left(\sum_1^n \frac{1}{t_i}\right)^2\right]^{-1/2} \quad (32)$$

Form a practical point of view, it is of interest to determine a lower natural tolerance limit (L_I) for durability [35] which implies the construction of the $L_I(t_1, t_2, \ldots, t_n)$ statistics, so that:

$$P\left\{\int_{L_I}^{\infty} \tilde{f}_t(t; \alpha, \beta) dt \geq P\right\} = \gamma \quad (33)$$

which is successively transformed in this way:

$$P\{\tilde{F}_t(\infty; \alpha, \beta) - F_t(L_I; \alpha, \beta) \geq P\} = \gamma \quad (34)$$

respectively:

$$P\left\{F_0\left(\frac{\beta}{L_I}-\alpha\right)\geq P\right\}=\gamma \quad (35)$$

and:

$$P\left\{\left(\frac{\beta}{L_I}-\alpha\right)\geq F_0^{-1}(P)\right\}=\gamma \quad (36)$$

at last:

$$P\left\{\frac{1}{\beta}\leq\frac{1}{\alpha+F_0^{-1}(P)L_I}\right\}=\gamma \quad (37)$$

As shown in (5), the maximum likelihood estimator for $1/\beta$ is distributed approximately normally, of class $N\left(\frac{\alpha}{\beta},\frac{1}{n\beta^2}\right)$, in which case (37) can be written:

$$P\left\{\frac{n^{-1}\sum_{i=1}^{n}t_i^{-1}-\frac{\alpha}{\beta}}{\beta^{-1}n^{-1/2}}\leq\frac{\frac{1}{\alpha+F_0^{-1}(P)L_I}-\frac{\alpha}{\beta}}{\beta^{-1}n^{-1/2}}\right\}=\gamma \quad (38)$$

in which the second part of the inequality (38) is γ, the standardized normal distribution quartile. Thus, the lower tolerance limit is:

$$L=\frac{1}{[\hat{\alpha}+F_0^{-1}(p)]\left(\frac{u_\gamma}{\hat{\beta}\sqrt{n}}+\frac{\hat{\alpha}}{\hat{\beta}}\right)} \quad (39)$$

5. RESULTS

In theory, in some cases, the proposed models for the behavior of different characteristics of product optimization, or modeling economic processes, do not find instantly or always correspond in the real world [36, 37, 38, 39, 40, 41]. However, this is a way to reduce and simplify the interaction between theory and practice. In most cases, the theory goes beyond the practical possibilities of its illustration and application. Thus, a statistical model, which describes the behavior at extreme demands, is the Alpha distribution, which has a theoretical justification and can illustrate a real case with the collected data.

The data presented (table 2) are collected by the testing laboratory of the Quality Assurance Department of a truck manufacturer and represent the results of the observation of the durability for a number of 324 helical drills with a diameter of ϕ20 mm, the data being expressed in minutes. Experimental data were recorded at the company Roman S. A. in Brasov [42]. Roman S.A. is a truck and bus manufacturer from Brasov, Romania. It also manufactures various components for trucks like engines, axles, decks etc. The data were systematized in the form of an interval distribution series. The calculations lead to the indicators presented in Table 3.

Durability distribution

Table 2

Interval (min)	Frequencies
under 75	18
75 - 100	43
100 - 125	69
125 - 150	89
150 - 200	39
200 - 250	32
250 - 350	21
over 350	13
Total	324

[1] *Sample data collected at Roman S. A.*

The results of the calculations indicate that the series has a strong asymmetry to the right: mode = 132.14 < median = 134.13 < mean = 155.59 indicates and alpha distribution. Based on this relation (26) the ratio between the modal value and the value of the median results: K = 0.985, and from table 1 the value of the parameter α = 11.35 is extracted, also the parameter β is determined by (19):

$$\hat{\beta} = \frac{\hat{\alpha}^2 \bar{t}}{1+\hat{\alpha}^2} = \frac{11.35^2 \cdot 155.59}{1+11.35^2} = 154.39$$

Descriptive Statistics

Table 3

Statistical indicators	Title 2
mean	155.59
variance	5851.71
standard deviation	76.50
coefficient of variation (%)	49.16
mode	132.14
median	134.13
coefficient of skewness	0.8414

[1] *Authors' own calculation based on sample empirical data.*

Based on the estimated $\hat{\alpha}$ and $\hat{\beta}$ values, the lower tolerance limit L_I can be determined for the durability of the devices in case of the alpha distribution, as follows:

$$P\left\{\int_{L_I}^{\infty} \tilde{f}_T(t; 11.35, 154.39)dt \geq 0.90)\right\} = 0.95$$

So, the indicators are: $P = 90\%$, $n = 324$, $\gamma = 0.95$, $u_\alpha = 1.65$, $\hat{\alpha} = 11.35$, $\hat{\beta} = 154.39$. Replacing in the relation (39) the calculations lead to: $L_I = 1.071$, thus practically, with a 95% probability, at most 10% of the population considered will have a durability of less than 1.071 minutes.

6. CONCLUSIONS

The economic impact of the durability of cutting tools is major on the efficiency of industrial enterprises with a mechanical profile, as they are essential in optimizing the cost of replacing processing tools with losses generated by the use of worn out devices. This situation of prolonged use of some damaged devices, would impact the deterioration of the components and in the end the loss of the quality of the products assembled from them.

The importance of this subject is also reflected in the multitude of scientific papers and expert meetings dedicated to metal cutting and finding the balance between costs and consequences. These concerns also include the identification of the most appropriate statistical models to describe and allow predictions on the behaviour of cutting devices to stress and extreme loads. This area of interest also includes the Alpha model, proposed in 1968 by the engineer G. V. Drujinin, a model that in our opinion did not have the expected impact in relation to the modelling qualities that this distribution law has.

The interest for this statistical distribution law does not only focus on the practical side, of solving problems strictly related to supply management, stocks of cutting tools and replacement moments, but also strictly theoretically, for the development of methods and procedures for this law, starting with the testing of statistical hypotheses up to the generalization of this function, establishing limits of natural tolerance, indicators with great practical utility, and up to designing methods based on survey data, etc. We consider that theoretical developments and practical destinations have not exhausted their knowledge resources and application destinations. Thus, from the point of view of the applicability of this distribution, a wide horizon opens for contributions regarding the modeling of resilience processes at the macroeconomic level, as a result of various crisis situations.

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