Measuring equilibrium models: a multivariate approach

Nadji RAHMANIA
Laboratoire de Probabilites et Statistique, USTL France

Abstract

This paper presents a multivariate methodology for obtaining measures of unobserved macroeconomic variables. The used procedure is the multivariate Hodrick-Prescot which depends on smoothing parameters. The choice of these parameters is crucial. Our approach is based on consistent estimators of these parameters, depending only on the observed data.

JEL classifications: E31, E32, C22, C32

Key words and phrases: detrending, Hodrick-Prescott filter, noise-to-signal ratio, Nairu, Potential output, unobserved components models.

1. Introduction

Contemporary literature often tries to present unobserved variable as an equilibrium value at which the observed or realised GDP values are compared. The output gap, which is commonly defined as the diference between actual and potential GDP, is a common way to assess the appropriateness of monetary and fiscal policies.

The recent literature presents three major methods to consider unobserved variables as potential GDP. The first one is the development of a structural model with theoretical based foundations. This gives potential growth estimation. The second method, which makes a significant use of statistical properties, consists of the application of a smoothing filter on the observed time series and gives a growth trend which is believed to be potential growth. The last method crosses the theoretical based models with statistical techniques methodology.

Among the wide range of procedures, the filter method implements statistical procedures to the selected variables. The univariate Hodrick-Prescott (HP hereafter) filter (Hodrick and Prescott 1980, 1997) is probably the most frequently adopted procedure, especially because of its simplicity. The HP filter which must be linked to Kalman filter is commonly used to estimate and predict business cycles and is applied by minimising residuals of one or more economic relations that involve at least the unobserved variable. This minimizing program is subject to a smoothness constraint α . The filtered series is a symmetrical and centred moving average, where the smoothing parameter is important because it conditions the smoothing extent of the observed time series. The smoothness of the series depends of the appropriate value of the positive value of smoothing parameter chosen by the user. Choice of smoothing parameter strongly conditions the qualitative behaviour of the filter, compared to observed series. A very high value of α will produce straight line trends estimates', and trend estimates that follow closely the data will be produce by low values of α . Some authors, (Baxter and King 1999, French 2001, King and Rebello 1993) show that the smoothing parameter value is subject to a strong uncertainty so that the definition of the smoothing parameter value appears defined in an ad hoc way, without real foundations, that can be linked to the observed series characteristics'. Numerous simulations' findings and recommended figures from standard econometrics and statistical packages usually recommend as smoothness constant, 100 for annual data, 1600 for quarterly data, and 14400 for monthly data.

Then, like a numerous univariate filtering methods, HP filter is purely historical and static. This HP filter becomes imprecise near the end of data. Indeed, the estimated trend is a moving average of the data where current estimate depends of future outcomes. The estimate carried out at the end of the sample does not include any more sufficient relevant observations to deduce an adequate current outcome because future observations are insufficient. HP filter cannot be used to conduct forecasts or to be a relevant instrument for evaluation of economic policy measures.

In order to improve the unobserved variable estimates, Laxton and Tetlow (1992) extend the standard HP filter by introducing economic constraints to the optimization problem associated with the HP filter. New equations will give informations about labour and good markets tensions. Then, their recently developed filtering method taking into account various economic variables gives a multivariate Hodrick-Prescott filter But, as univariate HP filter, last smoothing methods use ad-hoc defined standard smoothing parameter. Schlicht (2005) proposed an automatic two-sided filter method based on similar methods to estimate smoothing constant to define a non ad-hoc smoothing parameter. But mathematic convergence properties of this estimator are not demonstrated and consistency of Schlicht's filter remains unproved.

An original procedure for estimating potential output is presented that make use of new developed multivariate Hodrick-Prescott Filter (MVHP hereafter). New estimator of the smoothing parameter which is explicit and consistent, based on the optimality criterion firstly suggested by Schlicht (2005) is developed by Dermoune, Djehiche and Rahmania (2008,2009)(DDR hereafter). This new estimator is calculated starting from the whole set of available observations and does not depend any longer of the researcher's choices. This new estimator is also associated with a constructed interval confidence with a precise confidence level.

The main interest of this developed approach is that it is located in the middle ground of standard statistical approach of the time series univariate filtering method and use of a standard structural model. The purpose of this paper is to offer a rather comprehensive and new method to estimate of potential output using a multivariate Hodrick-Prescott Filter (MVHP) in order to make empirical estimates of macroeconomic time series. Main idea behind this MVHP is that more informations is considered from single times series data. This version of MVHP, which is developed by DDR (2008,2009) is an extension of Schlicht's suggestion.

This paper is organized as follows. Section 2 introduces the Hodrick-Prescott filter and Section 3 introduces a consistent Multivariat Hodrick-Prescott filter. Estimates of potential output for the United States economy are provided in section 4. A final section offers summary results, draws extensions of this empirical assessment and concludes.

2 The univariate Hodrick-Prescott filter

Let $x = (x_1, ..., x_T) \in \mathbb{R}^T$ be a time series of observables. The HP filter decomposes *x* into a nonstationary trend $y \in \mathbb{R}^T$ and a cyclical residual component (noise term) $u \in \mathbb{R}^T$:

$$x = y + u. \quad (2.1)$$

Given a smoothing parameter $\alpha > 0$, this decomposition of x is obtained by minimizing the weighted sum of squares

$$\|x - y\|^2 + \alpha\|D^2 y\|^2 \quad (2.2)$$

with respect to y, where for $a \in \mathbb{R}^T$, $\|a\|^2 = \sum_{i=1}^{T} a_i^2$.

Here, $D^2 y$ is the trend disturbance obtained by acting the second order forward shift operator D^2 on the trend $y = (y1, y2, ..., yT)$:

$D^2yt := (yt+2 - yt+1) - (yt+ 1 - yt), t = 1,2,..., T-2$, or, equivalently,

$$D^2 y_t := 2\left(\frac{y_{t+2}+y_t}{2} - y_{t+1}\right),$$

measuring the deviation between the value of the trend at $t+1$, y_{t+1} and the linear interpolation between yt and $yt+2$.

In vector form,

$$Py(t) = D^2 y_t, \qquad t = 1,\dots,T-2, \tag{2.3}$$

where, the shift operator P is the following (T+2) x T-matrix

$$P := \begin{pmatrix} 1 & -2 & 1 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ & \dots & \dots & \dots & \dots & \\ 0 & \dots & \dots & 1 & -2 & 1 \end{pmatrix}.$$

Following Schlicht (2006), a way to estimate the smoothing parameter α is to let the optimal solution *y(α,x)* in Eq.(2.2) be the best predictor of any trend *y* given the time series *x*, i.e.

$$y(\alpha,x) = E[y|x]. \tag{2.4}$$

This approach of estimating α assumes that we are able to compute explicitly this conditional expectation, which is not always the case. The normal and more generally the elliptical probability distributions are among the few models for which an explicit formula for the conditional expectation is possible. In order to estimate the trend and the smoothing parameter, given the time series of observations x, we obviously need a model for the joint distribution of *(x,y)*. This can be achieved through imposing a model for the joint distribution of *(u,v)*.

According to DDR(2008), under the assumption that the disturbances u and v are independent and normally distributed, the following statistics

$$\widehat{\alpha} = -\frac{1}{4}\left(\frac{3}{2} + \frac{(T-3)\sum_{j=1}^{T-2} Px(j)^2}{(T-2)\sum_{j=1}^{T-3} Px(j)Px(j+1)}\right)^{-1} \tag{2.5}$$

based on the time series of observation Px, is a consistent estimator of the smoothing parameter α.

The Hodrick-Prescott Multivariate filter

The MVHP seeks to estimate the unobserved variable y as a solution to the following minimization problem

$$\arg\min_{y}(||x-y||^2+\alpha_1||Py||^2+\alpha_2||\xi||^2), \tag{3.1}$$

given the following dynamics:

$$\begin{cases} x = y + u, \\ z := x^* - dX = \beta y + \xi, \\ Py = v, \end{cases} \tag{3.2}$$

where, x* is another explanantory variable and X is an exogenous variable affected by the parameter *d*. Moreover, a way to estimate the smoothing parameters α_1 and α_2 is to let the optimal solution y in (3.1) be the best predictor of any trend *y* given the time series (*x,z*), i.e.

y = E[*y*|*(x,z)*] (3.3)

DDR(2009), when the noise term u and the signal terms v are independent Gaussian vectors with mean zero, the following statistics:

$$\widehat{\alpha}_1 = -\frac{1}{4}\left(\frac{3}{2}+\frac{(T-3)\sum_{j=1}^{T-2}Px(j)^2}{(T-2)\sum_{j=1}^{T-3}Px(j)Px(j+1)}\right)^{-1},$$

$$\widehat{\alpha}_2 = \frac{\sum_{j=1}^{T-3}Px(j)Px(j+1)}{\sum_{j=1}^{T-3}Pz(j)Pz(j+1)},$$

and

$$\widehat{\beta} = \left(\frac{2(T-3)\sum_{j=1}^{T-2}Pz(j)^2+3(T-2)\sum_{j=1}^{T-3}Pz(j)Pz(j+1)}{2(T-3)\sum_{j=1}^{T-2}Px(j)^2+3(T-2)\sum_{j=1}^{T-3}Px(j)Px(j+1)}\right)^{1/2}$$

are consistent estimators for α_1, α_2 and β based only on the time series of observations *Px* and *Pz*.

Calibration

Empirical assessment refers to U.S. unemployment and U.S. GDP spanning from 1957:2 to 2009:2. Both times series are monthly seasonally adjusted, but are converted to quarterly frequency. The data for this empirical analysis is obtained from the Fed website database. We are focused on U.S. Consumer Price Index (CPI) and

U.S. unemployment rate. All items are about Consumer Price Index for All Urban Consumers, less Energy comes from Bureau of Labor Statistics of Department of Labor, and is provided by the bureau of labor statistics website. Unemployment rate is extracted from the same site and comes from the Bureau of Labor Statistics of U.S. Department of Labor.

In Tables 1, we collect the estimated values of the statistics α_1, α_2 and β based on Dermoune, Djehiche and Rahmania method.

$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}$
0.2372	0.4347	0.4486

Table 1: NAIRU: Estimated values of α_1, α_2 and β.

Figure 1 shows the pattern of the US unemployment rate over the period 1999:2 to 2009:2. The first curve represents the observed U.S. unemployment rate. The second curve fits the smooth trend which is obtained from a standard HP filter. Numerous simulations show that standard HP series will follow the data closely if the value of the smoothing coeffcient is low. A more important value, α = 1600 is the optimal smoothing parameter value usually adopted for quarterly data. This standard HP filter uses only the data itself to carry out the smoothing series and extracted the trend line. The third curve describes the U.S. unemployment rate trend which is estimated from the implementation of the multivariate HP filter as described by DDR (2009). Figure 1 clearly shows that unemployment rate series estimates according to this standard HP filter can be very different from the observed unemployment time series. Analysis of both Figure 1 and table 3 show that this standard HP filter will induce large unemployment gaps as given in Figure 3. The economic dynamism of the first half of 2000, which supports a significant reduction of unemployment in many economies and more particularly in U.S. economy over following years 2004-2006 are imperfectly analyzed by this standard HP filter. There would be an excessive unemployment compared to its trend over 1999:2-1999:4, 2001:4-2005:1 and 2008:4-2009:2 periods. These deviations are important and indicated the low capacity of the standard HP filter to establish forecasts. This

HP filter has low power to specify recent macroeconomic shocks as financial crisis and its incidence on

U.S. unemployment rate. Such weaknesses of standard HP filter are quite significant over the end of sample but not within sample.

Mechanical phenomenon and especially the lack of power for the border periods of filter are clearly highlighted in Figure 3 because gap between U.S. unemployment rate and HF filter are significant. Table 4 shows summary statistics for quarterly unemployment measures for U.S. economy for the period 1999:2-2009:2. It shows a low correlation between U.S. unemployment rate and HP filter because correlation is 0.70, whereas the correlation between the actual U.S. unemployment rate and MVHP is strong and high. Estimate of the latter correlation coeffcient between both series is 0.99. Those results, also shown by figure 1, describe powerful effects of MVHP over standard HP filter. Figure 4 which presents gaps between U.S. unemployment time series and HP filters confirms previous analysis because value of MVHP filter are less significant than standard HP values.

The HP filter does not appear relevant to evaluate the business cycles and their consequences on unemployment rates as well as unemployment gaps between both time series. Figure 1 and summary statistics from table 3 clearly support the idea that HP filter is not a powerful tool to establish forecasts, because it does not include enough additional economic relations and feedbacks with macroeconomic variables. The HP filter is not a relevant way to appreciate economic policy measures taken by national authorities.

The MVHP allows a larger consideration of the various economic variables leading to a low gap between MVHP and observed unemployment series inside series sample as well as at frontier values of sample. MVHP is clearly linked with

U.S. unemployment rate. This eases economic forecasts and the appreciation of economic policies measures. MVHP robustness compared to the HP filter results firstly from the fact that economic relations are taken onto consideration.

In Tables 2, we collect the estimated values of the statistics, α_1, α_2 and β based on DDR method.

$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}$
0.3899	51038	0.0622

Table 2: GDP: Estimated values of α_1, α_2 and β.

Figure 2 makes a comparison of U.S. GDP between actual U.S. GDP and both potential outputs. Results are quite different from previous calibration.

Important correlation between U.S. GDP time series is also presented in Table 3. But, those estimated correlations might appear different when assessing from a graph.

A visual inspection of figure 2 shows that the standard HP trend is linear over complete period. This graph indicates an actual GDP higher than the standard HP trend between 2003:3 and 2005:2 and over the recent period of 2008:4 to 2009:2. The trend of the standard HP filter is less powerful because the linear trend of recent period lasts whereas the GDP strongly reduces over this period. This standard HP filter does not constitute a powerful way to make GDP forecasts when there is an important unanticipated shock.

On the other side, comparison of actual GDP and MVHP shows close correlation. From MVHP time series, one can conclude that DDR (2009) is quite powerful to make adequate forecasts. Figure 4 makes comparison of two output gaps. Visual inspection of this graph clearly shows that gap between actual GDP and MV Potential GDP is always lesser than 1. Then, one can conclude that DDR (2009) method produces more significant results. This graph confirms coeffcient correlation figures from table 3, where correlation coeffcient between both time series is nearly 1. On the other hand, same graph shows that standard HP filter might appear larger than 1and last period's observations are clearly different from actual

U.S. GDP time series. Standard HP filter is not able to give close figures for border values of U.S. GDP, which is a common remark for this standard HP filter. This shows that this new estimator has more powerful effects than standard HP filter.

This table shows summary statistics for quarterly unemployment measures for U.S. economy for the period 1999:2-2009:2. This table shows a low correlation between

U.S. unemployment rate and HP filter because correlation is 0.73, whereas the correlation between the U.S. unemployment rate and HPMV is strong and high. Correlation coeficient between last coeficients is 0.99. This shows more powerful effects of HPMV on HP filter.

The HP filter appears relevant to evaluate the business cycles and their consequences on unemployment rates as well as unemployment gaps between both time seires. But graph 1 and summary statistics from table 1 clearly support the idea of HP filter is not a powerful tools to establish forecasts, because it does not include enough additional economic relations and feedbacks with macroeconomic variables. The HP filter is not a relevant way to appreciate economic policy measures taken by national authorities.

The HPMV allows a larger consideration of the various economic variables leading to a low gap between HPMV and observed unemployment series inside series sample as well as at frontier values of sample. HPMV is clearly linked with

U.S. unemployment rate. This eases economic forecasts and the appreciation of economic policies measures. HPMV robustness compared to the HP filter results firstly from the fact that economic relations are taken onto consideration.

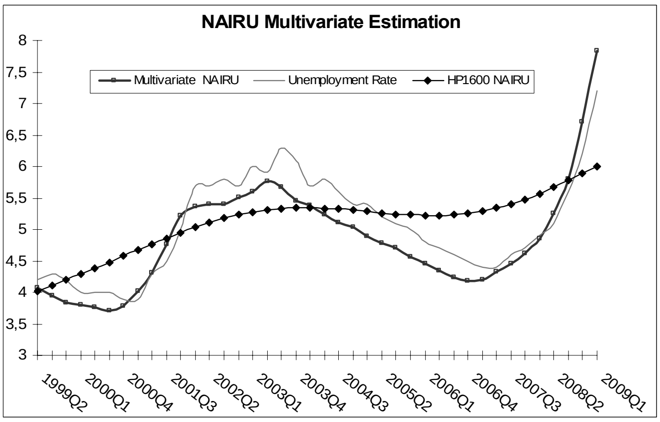
Conclusions

This paper presents results from a methodology based on the multivariate HP filter developed by Dermoune, Djehiche and Rahmania (2008, 2009). This methodology is implemented to U.S. unemployment rate and U.S. GDP data spanning from 1999:2 to 2009:2. This original methodology confirms that standard HP filter provides low powerful estimations for U.S. macroeconomic data, notably at the end of the sample, which should be most relevant for policymakers. Multivariate HP filter provides more realistic estimations for trend. This will then provide better references for macroeconomic forecasting. Those results confirm that this recent estimator is more powerful effects than standard HP filter.

Appendix

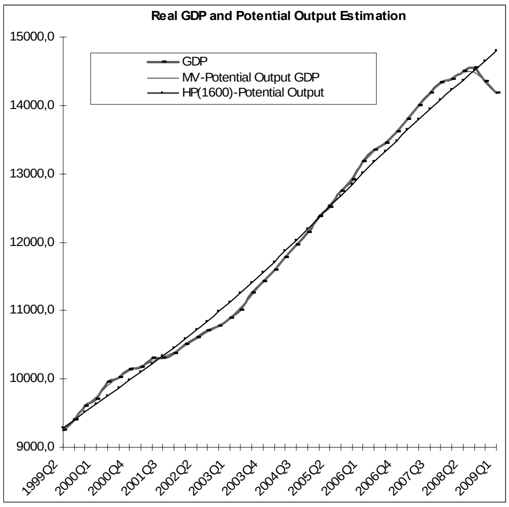
Nairu Multivariate Estimation

Figure 1



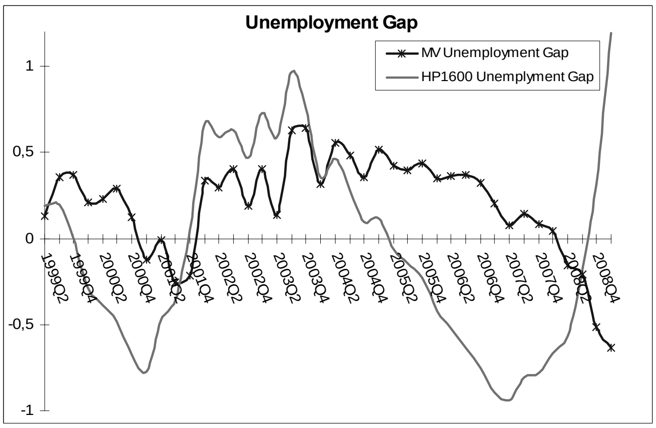
Potential Output Estimation

Figure 2



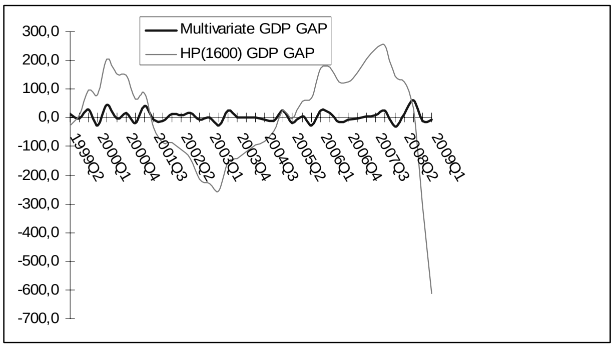
Unemployment Gap

Figure 3



Potential GDP Gap

Figure 4



Unemployment and GDP measures: summary statistics

Table 3

	Mean	Median	Min	Max	St.dev
Unempl.	5.06	5.00	3.9	7.2	0.79
HP(1600) Unempl.Trend	5.12	5.25	4.0	76.0	0.47
MVHP Unempl.Trend	4.86	4.77	3.71	7.83	0.85
GDP	4.07	4.06	3.96	4.16	0.06
HP(1600) GDP trend	4.07	4.07	3.96	4.17	0.06
MVHP GDP trend	4.07	4.06	3.96	4.16	0.06

Correlations

Table 4

	Unempl.	HP(1600) Unempl.Trend	MVHP Unempl.Trend
Unempl.	1		
HP(1600) Unempl.Trend	0.7	1	
MVHP Unempl.Trend	0.94	0.70	1
	GDP	HP(1600) GDP trend	MVHP GDP trend
GDP	1		
HP(1600) GDP trend	0.99	1	1
MVHP GDP trend	0.99	0.99	1

References

[1] Araujo, F. Areaso, M. B. M. and Neto, J. A. R. (2003): r-filters: a Hodrick-Prescott Filter generalization. Working Paper Series (69), Banco Central Do Brasil.

[2] Bell, L.(1984), Alternative method to estimate unobserved variables, OECD Economic Departement, Working Paper, 240.

[3] Boone, W.(2000), Signal extraction for nonstationary time series,The Annals of Statistics,12, 646-664.

[4] Chagny, O., Lemoine, M. (2003): Ecart de production dans la zone Euro: Une estimation par le filtre de Hodrick-Prescott multivarie, Revue de l'OFCE, Vol. 86, 173-196.

[5] Dermoune, A., Djehiche B., Rahmania N.(2008):A consistent estimator of the smoothing parameter in the Hodrick-Prescott filter. J. Japan Statist. Soc. ,Vol. 38(2),pp. 225-241.

[6] Dermoune, A., Djehiche B., Rahmania N.(2009): Multivariate Extension of the Hodrick-Prescott Filter-Optimality and Characterization. Studies in Nonlinear Dynamics and Econometrics, Vol. 13: No. 3, Article 4.

[7] Dermoune, A., Djehiche B., Rahmania N.(2009): A new measure of output gap, preprint, 16 pages.

[8] Hodrick, R. and Prescott, E. C. (1980): Postwar U.S. business cycles: An empirical investigation. Discussion paper 451, Carnegie-Mellon University.

[9] Hodrick, R. and Prescott, E. C. (1997): Postwar U.S. business cycles: An empirical investigation. Journal of Money, Credit and Banking. 29(1), 1-16.

[10] Kydland, E. F., Prescott, E. C. (1990): Business cycles: Real facts and monetary myth. Federal Reserve Bank of Minneapolis Quarterly Review, 14 (Spring), 3-18.

[11] Laxton, D., Tetlow, R.(1992), A simple multivariate filter for the measurement of potentiel output, Technical Report, 12, 35-45.

[12] Razzak, W. and Dennis, R. J. (1995): Estimates of New Zealand's output gap using the Hodrick-Prescott filter with a non-constant smoothing parameter. Discussion paper G95/8, Reserve Bank of New Zealand.

[13] Razzak, W. (1997): The Hodrick-Prescott technique: A smoother versus a filter. An application to New Zealand GDP. Economic Letters, (57), 163-168.

[14] Richardson P., Boone L., Giorne C., Meaci M., Rae D. , Turner D (2000) : The concept, policy use of structural unemployment: estimating a time varying Nairu across 21 OECD countries.Eco/WKP 23.

[15] Reeves, J. J., Blyth, C. M., Triggs, C. M. and Small, J. P. (2000): The Hodrick-Prescott Filter, a generalization and a new procedure for extracting an empirical cycle from a series. Studies in Nonlinear Dynamics and Econometrics. 4(1): 1-16.

[16] Schlicht, E. (2006): Estimating the smoothing parameter in the so-called Hodrick-Prescott filter. J. Japan Statist. Soc., Vol. 35 No. 1, 99-119.

[17] Schlicht, E. (1984): Seasonal adjustment in a stochastic model. J.Forecast., 25, 247-273.

[18] Trimbur, T.M . (2006): Detrending economic time series: A Bayesian generalization of the Hodrick-Prescott filter Statistische Hefte, 25, 1-12.