A LAPLACE TYPE PROBLEM FOR REGULAR LATTICES WITH CIRCULAR SECTION OBSTACLES

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Abstract

In this paper, we compute the probability that a segment of random position and of constant length intersects a side of a regular lattice with circular sections obstacles. In particular, we obtain the formula of a probability already computed by Caristi and Stoka, as well as the formula of the Laplace probability. The results can be used for possible applications in economy and engineering, in particular for transportation problems.

Keywords: geometric probability, stochastic geometry, random sets, random convex sets and integral geometry.

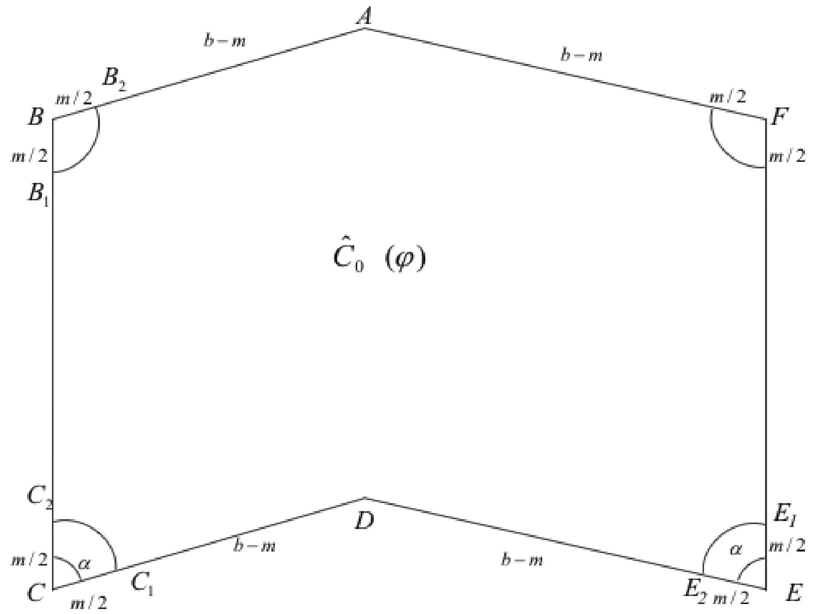
1. Introduction

In connection with some recent work (see [1], [2]), we consider a regular lattice with the fundamental cell represented in figure 1, and we compute the probability that a segment of random position and of constant length intersects a side of the lattice, i.e. the probability that the segment intersects a side of the fundamental cell. As particular cases of our study, we obtain the formula of a probability already computed in [1], and the formula of the Laplace probability.

2. Cells with circular section obstacles

Let $\Re(a,b,m;\alpha)$ be the regular lattice with the fundamental cell C_0 represented in Figure 1, where $m < \min(a,b)$ and $\alpha \le \frac{\pi}{2}$ is an angle. The obstacles are circular sections of two different types.

Fig. 1



We have:

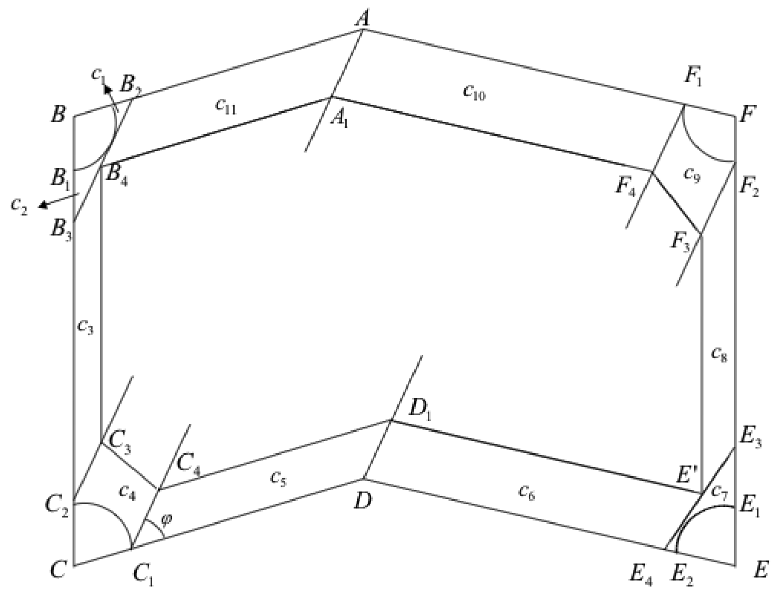
$$aria\, C_0 = 2ab\sin\alpha - \frac{\pi m^2}{4}. \quad (1)$$

We consider a segment s of random position and of constant length l with $l < \min\left(a - m, b - \frac{m}{2}\right)$, and we compute the probability P_{int} that this segment intersects a side of the lattice, i.e. the probability that the segment s intersects a side of the fundamental cell C_0.

The position of the segment s is determined by its middle point O and by the angle φ that the segment forms with the side *CD* of the fundamental cell C_0.

In order to compute the probability P_{int}, we consider the limit positions of the segment s for a fixed value of φ. Let $\hat{C}_0(\varphi)$ be the polygon obtained from these positions as in Figure 2:

Fig. 2

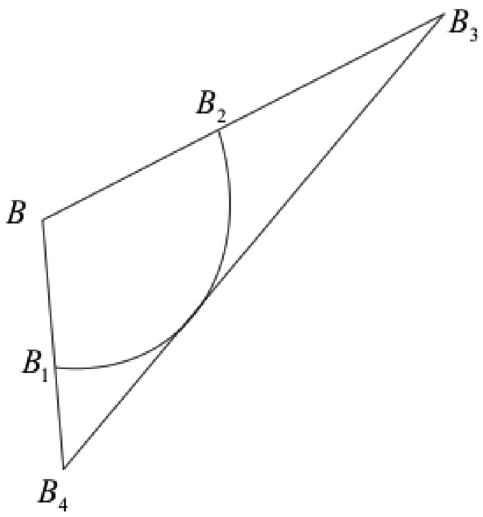


From figure 2, we may write:

$$aria\,\hat{C}_0(\varphi) = aria\,C_0 - \sum_{i=1}^{11} aria\,c_i(\varphi) \quad (2)$$

We consider the following figure:

Fig. 3



From

$$\textit{area circular section } BB_1B_2 = \frac{(\pi-\alpha)m^2}{8},$$

it follows that

$$\textit{area } c_1(\varphi) + \textit{area } c_2(\varphi) = \frac{l^2 \sin\varphi \sin(\alpha-\varphi)}{2\sin\alpha} - \frac{(\pi-\alpha)m^2}{8} \qquad (3)$$

We also have

$$\textit{area } c_3(\varphi) = \frac{l}{2}\sin(\alpha-\varphi)\left(a - \frac{m}{2} - \frac{l\sin\varphi}{\sin\alpha}\right). \quad (4)$$

Similarly, we have

$$\textit{area circular segment } C_1C_2 = \frac{m^2(\alpha-\sin\alpha)}{8}$$

and

$$\textit{area } c_4(\varphi) = \frac{ml}{2}\sin\frac{\alpha}{2}\cos\left(\frac{\alpha}{2}-\varphi\right) - \frac{m^2(\alpha-\sin\alpha)}{8}. \quad (5)$$

Replacing α with $\pi - \alpha$, we obtain

$$area\, c_9(\varphi) = \frac{ml}{2}\cos\frac{\alpha}{2}\sin\left(\frac{\alpha}{2}+\varphi\right) - \frac{m^2(\pi-\alpha-\sin\alpha)}{8}. \qquad (6)$$

We also have that

$$area\, circular\, section\, EE_1E_2 = \frac{\alpha m^2}{8}$$

and then

$$area\, c_7(\varphi) = \frac{l^2\sin(2\alpha-\varphi)\sin(\alpha-\varphi)}{2\sin\alpha} - \frac{\alpha m^2}{8} \qquad (7)$$

and

$$area\, c_5(\varphi) = \left(b - \frac{m}{2}\right)\frac{l}{2}\sin\varphi \,.(8)$$

Similarly, we obtain

$$area\, c_6(\varphi) = \left[b - \frac{l\sin(\alpha-\varphi)}{\sin\alpha}\right]\cdot\frac{l}{2}\sin(2\alpha-\varphi) \qquad (9)$$

and

$$area\, c_8(\varphi) = \left[a - \frac{m}{2} - \frac{l\sin(2\alpha-\varphi)}{\sin\alpha}\right]\cdot\frac{l}{2}\sin(\alpha-\varphi). \quad (10)$$

Also,

$$area\, c_{10}(\varphi) = \left(b - \frac{m}{2}\right)\cdot\frac{l}{2}\sin(2\alpha-\varphi) \qquad (11)$$

and

$$area\, c_{11}(\varphi) = \left[b - \frac{l\sin(\alpha-\varphi)}{\sin\alpha}\right]\cdot\frac{l}{2}\sin\varphi\,. \qquad (12)$$

Replacing in (2) the expressions (3), (4), (5), (6), (7), (8), (9), (10), (11) and (12), we obtain:

$$area\,\hat{C}_0(\varphi) = area\,C_0 - \left\{ \frac{l^2 \sin\varphi \sin(\alpha-\varphi)}{2\sin\alpha} - \frac{(\pi-\alpha)m^2}{8} + \right.$$

$$+\left(a - \frac{m}{2} - \frac{l\sin\varphi}{\sin\alpha}\right)\frac{l}{2}\sin(\alpha-\varphi) +$$

$$+\frac{ml}{2}\sin\frac{\alpha}{2}\cos\left(\frac{\alpha}{2}-\varphi\right) - \frac{m^2(\alpha-\sin\alpha)}{8} +$$

$$+\frac{ml}{2}\cos\frac{\alpha}{2}\sin\left(\frac{\alpha}{2}+\varphi\right) - \frac{m^2(\pi-\alpha-\sin\alpha)}{8} +$$

$$+\frac{l^2\sin(2\alpha-\varphi)\sin(\alpha-\varphi)}{2\sin\alpha} - \frac{\alpha m^2}{8} +$$

$$+\left(b-\frac{m}{2}\right)\frac{l}{2}\sin\varphi + \left[b - \frac{l\sin(\alpha-\varphi)}{\sin\alpha}\right]\cdot\frac{l}{2}\sin(2\alpha-\varphi) +$$

$$+\left[a - \frac{m}{2} - \frac{l\sin(2\alpha-\varphi)}{\sin\alpha}\right]\cdot\frac{l}{2}\sin(\alpha-\varphi) +$$

$$+\left(b-\frac{m}{2}\right)\cdot\frac{l}{2}\sin(2\alpha-\varphi) +$$

$$\left. +\left[b - \frac{l\sin(\alpha-\varphi)}{\sin\alpha}\right]\cdot\frac{l}{2}\sin\varphi \right\} =$$

$$= area\,C_0 - \left\{ \frac{l}{2}\left[2a\sin\alpha + \left(2b - \frac{m}{2}\right)\sin 2\alpha\right]\cos\varphi + \right.$$

$$\frac{l}{2}\left[2b + \frac{m}{2} - (2a-m)\cos\alpha - \left(2b - \frac{m}{2}\right)\cos 2\alpha\right]$$

$$\left. \sin\alpha - \frac{l^2}{2}\sin(2\alpha-\varphi) - \frac{m^2}{2}(\pi - \sin\alpha) \right\}. \qquad (13)$$

We denote by M the set of segments s that have the middle point in the fundamental cell C_0 and by N the set of segments s completely contained in the fundamental cell C_0. Then we may write (see [4]):

$$P_{int} = 1 - \frac{\mu(N)}{\mu(M)}, \qquad (14)$$

where μ is the Lebesgue measure in the Euclidean plane. We compute the measures $\mu(M)$ and $\mu(N)$ by using the kinematic measure of Poincaré (see [3]). If x and y are the coordinates of the middle point of the segment s and φ is the angle from above, then we may write

$$\mu(M) = \int_0 d\varphi \iint_{\{(x,y)\in C_0\}} dxdy = \int_0 [area\, C_0]d\varphi = \alpha \cdot area\, C_0 \qquad (15)$$

and

$$\mu(N) = \int_0^\alpha d\varphi \iint_{\{(x,y)\in \hat{C}_0(\varphi)\}} dxdy = \int_0^\alpha \left[area\, \hat{C}_0(\varphi)\right] d\varphi =$$

$$= \alpha \cdot area\, C_0 - \left\{ \frac{l}{2}\left[2a\sin\alpha + \left(2b - \frac{m}{2}\right)\sin 2\alpha \right]\sin\varphi - \right.$$

$$\frac{l}{2}\left[2b + \frac{m}{2} - (2a - m)\cos\alpha - \left(2b - \frac{m}{2}\right)\cos 2\alpha \right] \cdot$$

$$\left. \cos\varphi - \frac{l^2}{4}\cos(2\alpha - \varphi) - \frac{m^2}{2}(\pi - \sin\alpha)\varphi \right\}\Bigg|_0^\alpha =$$

$$= \alpha \cdot area\, C_0 - \left(\left\{ \left[2a\sin\alpha + \left(2b - \frac{m}{2}\right)\sin 2\alpha \right]\sin\alpha + \right.\right.$$

$$+ \left[2b + \frac{m}{2} - (2a - m)\cos\alpha - \left(2b - \frac{m}{2}\cos 2\alpha\right) \right]$$

$$\left.\left. (1 - \cos\alpha) \right\} \frac{l}{2} - \frac{1 - \cos 2\alpha}{4} l^2 - \frac{\alpha(\pi - \sin\alpha)m^2}{2} \right). \qquad (16)$$

From (2), (14), (15) and (16), we obtain

$$P_{int} = \frac{1}{\alpha\left(2ab\sin\alpha - \frac{\pi m^2}{4}\right)}\left(\left\{\left[2a\sin\alpha + \left(2b - \frac{m}{2}\right)\sin 2\alpha\right]\sin\alpha + \right.\right.$$

$$\left. + \left[2b + \frac{m}{2} - (2a-m)\cos\alpha - \left(2b - \frac{m}{2}\cos 2\alpha\right)\right](1-\cos\alpha)\right\}$$

$$\left.\frac{l}{2} - \frac{1-\cos 2\alpha}{4}l^2 - \frac{\alpha(\pi - \sin\alpha)m^2}{2}\right). (17)$$

For $\alpha = \frac{\pi}{2}$, the fundamental cell becomes a rectangle with sides a and $2b$ and with four quarter-circles of radius $\frac{m}{2}$. In this case, the probability (17) becomes:

$$P = \frac{2(a+2b)l - l^2 - \frac{\pi(\pi-1)m^2}{2}}{\pi\left(2ab - \frac{\pi m^2}{4}\right)}, \quad ,$$

a formula already obtained in [1]. Moreover, for $m \to 0$, we obtain the formula of the Laplace probability:

$$P = \frac{2(a+2b)l - l^2}{2\pi ab}.$$

References

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