MODELING UNDER CONTINUOUS AND DISCRETE TIME CONDITIONS

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Abstract

Modeling for portfolios is a very delicate process. There are enough mountain theories that prove one way or another of action. However, in establishing an apt-to-use model, a study based on Brownian evolution that defines the process of short-term interest rates must first be conducted. Then, after clarifying this aspect, it is necessary to approximate the variables that enter the model, under continuous time and discrete time conditions.

We can also use the terms discrete time or continuous time, these having the same meaning. The discrete time approximation can be done by the Euler method or by the local linearization method. The article expresses the concrete situations for the three possibilities mentioned above. The study also addresses the empirical results regarding the modeling of short-term interest rates.

Of course, to model short-term interest rates, Euler or Milestein approximations must be applied for short-term data. Taking an example, it is specified what their evolution over time is. Also, it should be noted that in the case of these models, the verification of the autocorrelation, the normality test and the comparison with the results of the Kerls estimation model must be carefully checked. The respective data are presented sequentially and ensure an understanding of the phenomenon to which we refer.

Keywords: *continuous time, discrete time, models, tests, coefficients, significance, evolutions, forecast*

JEL Classification: C10, E43

Introduction

Switching from a continuous time model to a discrete time model is not an easy problem. We will approach various procedures to transform continuous time models into discrete time models. There are many ways to convert continuous time patterns into discrete time variants. In this sense we mention the Euler method, the Milstein method as linearization methods.

The generic problem is valid for a thorough study in the field of empirical finance. We will analyze the methodological problems regarding the modeling of data on the short-term interest rate. Short-term interest movements have become essential for asset price theories, bond prices and long-term structure studies. The value of derivatives, risk neutral assessments and dynamic portfolio theory, but also for macroeconomic behavior, such as consumer and investment behavior and exchange rates, are equally important and sensitive. That is why we will focus on modeling the short-term interest process.

After some methodological explorations, we will try by a few examples to estimate short-term interest rates. The results of the specification tests of the autocorrelation and normality indicate that the procedure followed is complete and subject to risks. We know that the ARMA-ARCH model with level-dependent volatility performs better than the continuous-time models in terms of probabilistic values, specification tests, as well as the forecasts in the sample and beyond.

Literature review

Anghelache and Anghel (2019) address in their work the problems related to the modeling of various economic phenomena through case studies in accordance with economic theory. Anghelache and Anghel (2018) studies and presents the results of case studies, analyzed by econometric modeling methods of economic phenomena. Anghelache and Anghel (2016) approaches the theoretical concepts of econometric modeling of economic phenomena. Anghelache (2012) analyzes the main aspects of economic modeling. Brenner et al. (1996) address in their work the problem of dynamic allocation of assets under inflation. Chan et al. (1992) presents an empirical comparison of alternative models of the short-term interest rate. Cox et al. (1985a) presents a theory of the structure that characterizes interest rates. Newbold, Karlson and Thorne (2010) present a number of issues related to business statistics. Turnovsky (2000) studies the main methods used in macroeconomics. Vasicek (1977) analyzes and characterizes the balance of interest rate structure terms.

Methodology, data, results and discussions

• Study on evolution over time

We will initially consider a Brownian evolution that defines a process of short-term interest rates. In general, a Brownian process is the solution of a stochastic differential equation (SDE) of form:

$$dX_t = b(X_t, \theta)dt + a(X_t, \theta)dW_t \tag{1}$$

where $(dW_t)_{t\geq 0}$ it's a Brownian movement. In modern finance theory, diffusion processes are often used to model data from financial time series, such as the short-term interest rate. The short-term interest rate is important in characterizing the term structure of interest rates, which means the structure of interest rates with different maturities, as well as in creating prices at contingent interest rate applications. Some authors Vasicek (1977) or Cox et al. (1985a) are pioneers in the field. A study of some works is presented in Chan et al. (CKLS, 1992). Chan et al. (1992) show that many models with a factor for short rates can be synthesized through the relationship:

$$dX_t = (c - \beta X_t)dt + \sigma X_t^{\gamma} dW_t \tag{2}$$

The characteristic of this equation is that it has a coefficient of derivation in mean-inversion and a diffusion coefficient dependent on level. Such a continuous-time framework, although it may provide theoretical elements, implies some difficulties in empirical research. The first problem is how to estimate the parameters of this continuous model. There are many methods and attempts to implement the estimates. Among them we mention the method of indirect inference, the approximate method of probability, the general method of the moment regarding the diffusion generators, the efficient method of the moment, the nonparametric method, the method of approximation of density, which approached the new local linearization, suggesting and formalizing a series of concrete modeling possibilities.

The second problem of continuous time modeling is to judge the specifications of the model used in relation to the empirical data. For example, specification tests for diffusion processes may be offered. We will use three transformation methods so that we can solve both problems simultaneously. The three transformation methods are the Euler method, the Milstein method and the NLL method. These three methods provide discrete-time models for the discrete-time observed data of a diffusion process.

We can implement the maximum likelihood estimation and the prediction is quite easy, using these mentioned models. To test the model specifications of the three models with discrete time, we follow the following strategy. Using discrete-time approximations, we can easily transform the economic time series into a white noise process, which is normally distributed. Therefore, we test whether the estimated white noise for each approximate discrete-time model is normally distributed. The intuition is that if the discrete-time approximation correctly represents the data generation process, then we can eliminate the whole deterministic structure. The more obvious the deterministic structure, the better we can predict the data. We will compare the performance of the three approaches in discrete time.

The Euler approach is the simplest and most common discrete-time approximation. Its disadvantage is that the Euler estimator is not consistent. For better results, Milstein and NLL approximations are suggested. The improvement of these approximations is represented by smaller errors of parameter estimates in numerical experiments.

The evaluation of approximate models in discrete time, besides taking into account the accuracy of parameter estimation, requires the accuracy of the prediction. If the coefficient is linear, we find that the Euler and NLL approximations are equivalent in reparameterization, having the same predictor. We can deduce a functional relationship between the Euler approximation estimation and the NLL approximation estimation. Using this relationship we can highlight the better approximation performance.

In the numerical experiment, we do not have to consider the NLL method. We only compare the Euler and Milstein approximations of the numerical experiment using Monte-Carlo simulations. The results do not verify the superiority of the Milstein approximation over the Euler approximation. Parameter estimates and one-step predictions of the two models are similar, due to the small dimensions of the deviation parameters. The small dimension of the parameters has the same effect as the small discretization steps, because the observed variable evolves less in the case of smaller parameters and for shorter evolution intervals.

We know that if the discretization stages are small, then the effect of discretization is also small. The reason for employing small parameters is due to the fact that they are suggested by the empirical results of the data. Considering the specification of the model, we determine the estimated white noise of the Euler and Milstein models. The estimated white noise of the two approaches passes the specification test in most simulations. Therefore, the approximate models Euler and Milstein can correctly identify the deterministic structure of the real data. We also find that the Milstein approximation reduces the continuous-time effect better than the Euler approximation with respect to the estimated white noise distribution. We observe that for a high frequency the rejection of the Milstein method distribution test is lower than for the Euler method. In addition to the numerical experiment, we apply the approximate models of Euler and Milstein to short-term interest rate data. As in the numerical experiment, we implement ML estimation, one-step prediction, and test the model specification. Two approximate models work in the same way. The results here indicate a significant difference between simulated and real data: none of the short-rate data can pass the specification test satisfactorily.

There is a need for white noise and, as a consequence, we need to identify new models that explain the autocorrelation. Data simulated by those models with continuous time cannot accurately explain the high autocorrelation of white noise.

Since we cannot find a suitable model in continuous time, we call the discrete time. We use the autoregressive model to express the high self-correlations of the estimated white noise. We find that we can model the self-correlation of the estimated noise, taking more delays in the models. By summarizing the two modeling strategies, we use the ARMA-ARCH model with level dependent volatility. The model generalizes that of Brenner et al. (1996) by using the ARMA structure. Note that the approximate models Euler and NLL are equivalent in reparameterization. The Euler and Milstein approximations will be applied to real data for a short period. We can use the ARMA-ARCH model with level-dependent volatility to model short-term interest rates.

• Discrete-time approximation models

The difficulty of estimating maximum likelihood (ML) based on discrete-time observation is well known in the literature. In the following, we use approximate models in discrete time, so that ML estimation, prediction and model specification test are feasible. We present briefly the three methods of discrete-time approximation: Euler, Milstein and the new local linearization method (NLL).

- The Euler method

The idea of the Euler method is to replace dt with a time interval δt to obtain a discrete time approximation for the diffusion process X:

$$X_{t_{i+1}} - X_{t_i} = b\left(X_{t_i}, \theta\right)\Delta t_i + a\left(X_{t_i}, \theta\right)\Delta W_{t_i} \tag{3}$$

- Milstein Method

The Milstein method approximates SDE by the following relation:

$$X_{t_{i+1}} - X_{t_i} = b\left(X_{t_i}, \theta\right)\Delta t_i + a\left(X_{t_i}, \theta\right)\Delta W_{t_i} + \frac{1}{2} a\left(X_{t_i}\right) a'\left(X_{t_i}\right)\left(\left(W_{t_i}\right)^2 - \Delta t_i\right) \tag{4}$$

where $\Delta t_i = (t_{i+1} - t_i)$ și $\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$ Major expansion of the convergence order 1.0. It also has an additional term than the Euler method, which is the extension of the convergence order 0.5.

We have to make two observations:

a) As mentioned, we can apply Taylor expansion of different convergence orders to obtain different approximations in discrete time for the diffusion process. Therefore, these models are usually used for simulation, but not for estimation. Using such discrete-time models for estimating maximum likelihood, their density functions are complicated, and maximizing the likelihood function is unstable.

b) The Milstein method is just a better simulation method for diffusion processes when the simulation step size tends to zero. If the steps are fixed by the observation times {t_0, t_1, ..., t_N}, as in the present case, then we cannot emphasize the superiority of the Milstein method.

- The new local method of linearization

The new local linearization method (NLL) is based on the following idea: the Euler method keeps the drift and the diffusion coefficients for $s \in [t_i, t_{i+1})$, while the NLL approximates the deviation coefficient $b(X_s)$ until the second order terms using the formula:

$$dX_s = \left(b(X_{t_i}) + b'(X_{t_i})(X_s - X_{t_i}) + \frac{1}{2}b''(X_{t_i})a^2(X_{t_i})(s - t_i)\right)ds + a(X_{t_i})dW_s \qquad (5)$$

The diffusion coefficient is kept as a constant. Equation (5) can be solved analytically, the solution being given at t_{i+1} according to the relationship:

$$X_{t_{i+1}} - X_{t_i} = \frac{b(X_{t_i})}{b'(X_{t_i})}\left(e^{b'(X_{t_i})(t_{i+1}-t_i)} - 1\right) + \frac{b''(X_{t_i})}{\left(b'(X_{t_i})\right)^2}\frac{a(X_{t_i})^2}{2}\left(e^{b'(X_{t_i})(t_{i+1}-t_i)} - 1 - b'(X_{t_i})(t_{i+1} - t_i)\right) + a(X_{t_i})\int_{t_i}^{t_{i+1}} e^{b'(X_{t_i})(t_{i+1}-z)}\, dW_z \qquad (6)$$

The distribution of the last term can be done according to the relationship:

$$a(X_{t_i})\int_{t_i}^{t_{i+1}} e^{b'(X_{t_i})(t_{i+1}-z)}\, dW_z \sim N\left(0, a(X_{t_i})^2\int_{t_i}^{t_{i+1}} e^{2b'(X_{t_i})(t_{i+1}-z)}\, dz\right) \qquad (7)$$

We will show that Euler and NLL predictors of SDE (2) are equivalent. The reason is the linearity of the deviation coefficient from equation (2). We can easily see that the Euler approximation in the relationship:

$$X_{(i+1)\Delta t} - X_{i\Delta t} = (c - \beta X_{i\Delta t})\Delta t + \sigma X^{\gamma}_{i\Delta t} \Delta W_{i\Delta t} \tag{8}$$

and the NLL approximation, given by the relationship:

$$X_{(i+1)\Delta t} - X_{i\Delta t} = \frac{h_1(\beta)}{\beta}(c - \beta X_{i\Delta t})\Delta t + \sigma X^{\gamma}_{i\Delta t} \Delta W_{i\Delta t} \tag{9}$$

are equivalent through reparameterization, because:

$$\beta_{eu}\Delta t = h_1(\beta_{nll}) = 1 - e^{-\beta_{nll}\Delta t}$$
$$c_{eu}\Delta t = \frac{c_{nll}}{\beta_{nll}} h_1(\beta_{nll})$$
$$\gamma_{eu} = \gamma_{nll}$$
$$\sigma_{eu} = \sigma_{nll} h_2(\beta_{nll}) = \sigma_{nll}\sqrt{\frac{1-e^{-2\beta_{nll}\Delta t}}{2\beta_{nll}\Delta t}} \tag{10}$$

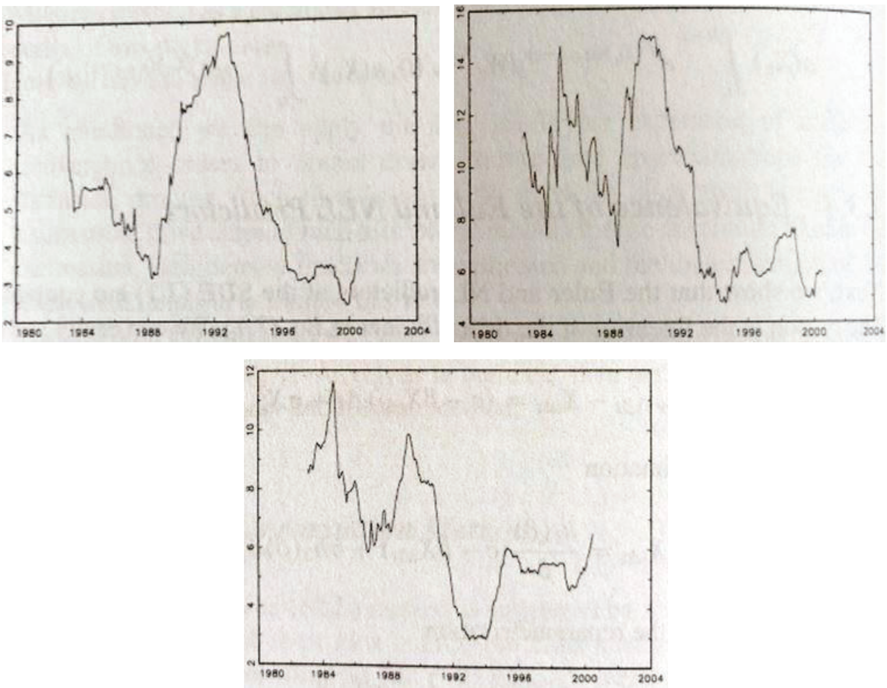
where $U_i, i = 1, \ldots$ are $N(0, \Delta t)$ distributed.

- **Empirical results regarding the modeling of short-term interest rates**

To model short-term interest rates, we apply Euler and Milstein approximations to short-term data. The hourly series of the money flow rate is shown in Figure 1:

Interbank rate based on economic indicators

Figure 1



• Application of specification tests

The main idea of the specification tests is to check if there is a deterministic structure in the residues. We apply two specification tests. The first test is to check if the residues are automatically correlated, and the second is to test whether the residues have white noise.

- Checking the autocorrelation

Whether $U_i, \dots, U_N$ identically distributed, random variables, with $E[U_i] = 0$, $Var[U_i] = 1$ și $E|U_i|^s < \infty$, to all $s \geq 2$. Fie R, the autovariance function given by the relation:

$$\hat{R}_k = \frac{1}{N-k} \sum_{i=k+1}^{N} U_i U_{i-k} \tag{11}$$

where we have $E[\hat{R}_k] = 0$ și $Var[\hat{R}_k] = \frac{1}{N-k}$, for $k \geq 1$

$$\hat{r}_k = \frac{\hat{R}_k - E[\hat{R}_k]}{\sqrt{Var[\hat{R}_k]}} = \sqrt{N-k}\hat{R}_k = \frac{1}{\sqrt{N-k}}\sum_{i=k+1}^{N} U_i U_{i-k} \tag{12}$$

Considering the sequence $(U_i U_{i-k})_{i=k+1,\dots,N}$ for a fixed k, this is close to that dependent on $(U_i)_{i=1,\dots,N}$. Central usage limits the above theorem because $\hat{r}_k$ converges to $N(0,1)$ in distribution as $N \to \infty$. Applying the test for discrete time approximations, we leave $U_i = W_i - W_{i-1}$. We should note that $\hat{r}_k \sim N(0,1)$ namely $\hat{R}_k \sim N(\frac{1}{N-k})$, is similar to the result $Var[\hat{R}_k] = \frac{1}{N}$ when N is large enough.

- Testing normality

We will use χ^2 for the histogram to test whether the distribution of the samples is a distribution $N(0,1)$. The idea is to compare the relative frequency of the samples at intervals I_m

$$\hat{p}_m = \frac{număr\ din\ \{i; U \in I_m\}}{N} \tag{13}$$

and $\hat{p}_m$ probability of distribution $N(0,1)$ on intervals I_m, where $\{I_m, m = 1, \dots, M\}$ are disjoint intervals of the real line. Weighted distance

$$d = \sum_{m=1}^{M} \frac{N}{p_m(1-p_m)} (\hat{p}_m - p_m)^2 \tag{14}$$

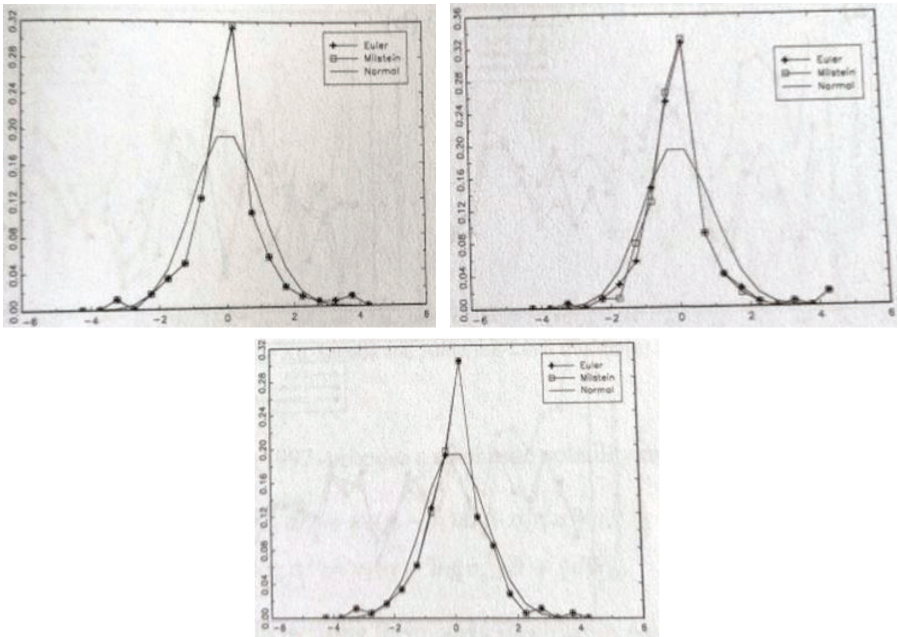
measures the distance between the sample and the normal distributions and converges to: $\chi^2(M-1)$, for distribution $N \to \infty$.

• Results of the CKLS estimation model

Parameter notations are changed because we will consider more general models. The estimation parameters, statistical for estimates and predictions are similar. We can also observe that the estimated deviation coefficients do not differ significantly from zero. If these are zero, it means that we cannot get better forecast values than using the current data. To see if it is possible to reduce the forecast error by using the models, the solution is to compare the forecast errors of the models with those of the primary forecast using the current data. Therefore, the results from the tables show that there is no evidence of such a model. The estimated white noise of the two approaches is also similar and is shown in Figure 2.

Distribution of estimated noise over time

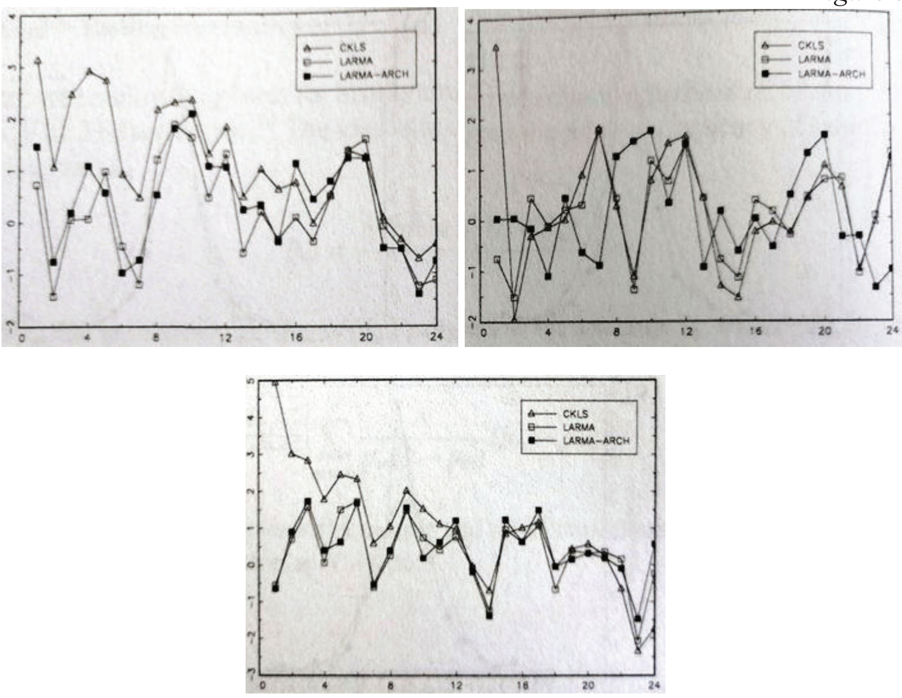
Figure 2



It can be observed that the estimated white noise is more concentrated around zero than the normal distribution and has residues. We also draw the normalized autocorrelations given in formula (12) for the Euler approximation of figure 3.

Normalized autocorrelation of the estimated noise

Figure 3



As a reference element, we present a Monte Carlo simulation for 1000 repetitions, using the estimation result. Most of them, namely 96%, have normalized maximum autocorrelations less than 2.8; considering the first ten normalized autocorrelations, and the maximum is only 4.2.

Values χ^2- test and their *p*-values are summarized in the figures 1; 2 și 3.

Conclusions

From the study carried out on modeling under continuous and discrete time conditions, a number of conclusions are drawn. The variables in the capital market, portfolios, assets, etc., can be studied under continuous time conditions, that is, an uninterrupted evolution or under discrete time conditions.

Some conclusions are drawn from the study carried out, namely, firstly, an approximation of the evolution of the factorial variables in continuous and in discrete time.

For approximation, Euler, Milstein method or local linearization method can be used. They help to approximate the variables considered, which introduced in this type of model under continuous or linearized time conditions give the required effects.

From the interpretation of the results on the short-term interest rate, it is concluded that in order to model the short-term interest rates, Euler and Milstein approximations must be applied to data with short rates expressed over time.

Of course, the main idea is that the specification tests, the autocorrelation verification and the normality test must be applied, because only in these conditions the results obtained are convincing and can be used in the studies and decisions taken in the field of portfolios in the capital market.

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