Model of Static Portfolio Choices

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Abstract

In decentralised economies the financial markets has a key role, being considered as institutions that transfer entrepreneurial risk to consumers. The entrepreneurial risk is assumed by the investors as part of the industrial or infrastructure investment that could be considered as the engine of the economic growth. The risk related to the investments finally is transferred from the investors to the tax paying population which statistically can be considered as risk-averse. The problem of the investors is to determine the optimum balance between the assumed risk and the expected performance, but having a limited investment capital.

In this paper we examined a simple version of the problem convincing risk-averse people to accept the purchase of risky assets by receiving an additional premium on it. Also, we focus on behaviour of investors who spend the entire investment at the end of the analysed period, but for simplicity we detach the time component of the equation.

Key words: *portfolio, asset, choice, model, risk*

Description of the One-Risky One-Risk-Free Asset Model

For the simple understanding of the model we consider an investor (also called agent) who has the capital to invest w_0. He can invest part of it (noted with α) in risky assets (e.g. mix of stocks) having the return over the period expressed as random variable $\tilde{x}$ and another part (noted with $w_0 - \alpha$) in risk-free assets (e.g. government bonds) with the return over the period r. The investor is interested in maximising the return at the end of period determined by the optimal composition (α, $w_0 - \alpha$) of his portfolio, which can be written as:

$$(w_0 - \alpha)(1 + r) + \alpha(1 + \tilde{x}) = w_0(1 + r) + \alpha(\tilde{x} - r) = w + \alpha\tilde{y}, \quad (1)$$

where $w = w_0(1 + r)$ is the final value invested in the risk-free portfolio, and $\tilde{y} = \tilde{x} - r$ can be described as “excess return” on the risky assets. The problem of investor is to choose α in that way that the Expected Utility (EU) should be maximised:

$$\alpha^* \in \arg\max_{\alpha} Eu\,(w + \alpha\tilde{y}) \quad (2)$$

We can interpret α = 0 as 100% investment in risk-free portfolio, and as α increase, the share of the risky portfolio increases and consequently the investor could accept higher exposure to the risk because of the higher expected net pay-off (earning).

Proposition 1.

Consider problem (1) where α^* is the optimal investment in the risky assets, and $\tilde{y}$ is the excess return of the risky assets over the risk-free rate. The optimal investment in the risky asset is positive if and only if the expected excess return is positive, meaning $\alpha^* E\tilde{y} > 0$. Moreover variation of α^* can be interpreted as:

(i) if α^* is decreasing, the risk-aversion of the investor is increasing in the sense of Arrow and Pratt;

(ii) if α^* is increasing than the risk-aversion is decreasing, meaning the investor can accept higher risk.

This is a simplified model which nod details the case why large proportion of population is risk-averse and does not hold any shares of stocks. This might be explained with the fact that obtaining information about the market evolution needs some additional knowledge, and obtaining such knowledge involve personal effort or cost, cost which the (private) investor could consider to high compared with the expected net earnings.

Another conclusion we can formulate is related to the fact that risk-averse people hold less risky portfolios, whereas rich people has decreasing risk-aversion and holds larger amount of stocks. Several empirical studies confirm this positive correlation between stock holding and wealth, which offers additional argument in favour of DARA.

In a particular case, let assume that the utility function show constant relative risk aversion:

$u(c) = \frac{1}{1-y}\, c^{1-y}$ for all c, where y is the degree of relative risk aversion.

In this condition the problem (2) can be written as:

$$E[\tilde{y}u'\,(w + \alpha^*\tilde{y})] = E[\tilde{y}(w + \alpha^*\tilde{y})^{-y}] = 0 \quad (3)$$

We can observe that the solution of this equation is $\alpha^* = kw$, where k is a positive constant, such that $E\tilde{y}(1 + k\tilde{y})^{-y} = 0$, leading to the conclusion that under constant relative risk aversion, the optimal amount of investment in risky assets is proportional to wealth.

Proposition 2.

Under constant relative risk aversion, the investors' willingness to invest in stocks is proportional to their wealth: $\alpha^*(w) = kw$.

We approach an approximate solution for determining the optimal demand by using first-order Taylor approximation to $u'(w + \alpha^* y)$ around *w*, we can approximate $E\tilde{y}u'(w + \alpha^*\tilde{y}) = 0$ as

$$E\tilde{y}[u'(w) + \alpha^*\tilde{y}u''(w)] \simeq 0$$

Finally, for the proportion of wealth invested in stocks, we obtain the approximation:

$$\frac{\alpha^*}{w} \simeq \frac{\mu_{\tilde{y}}}{\sigma^2_{\tilde{y}}} \frac{1}{R(w)}, \qquad (4)$$

where $R(w) = -wu''(w)/u'(w)$ is the relative risk aversion degree evaluated in *w*, and $\mu_{\tilde{y}}$ and $\sigma^2_{\tilde{y}}$ are respectively the mean and variance of the "excess stock" return. We can conclude, that the optimal share of wealth invested in stock is relatively proportional to the equity premium $\mu_{\tilde{y}}$ and inversely proportional to the variance of stock return and relative risk-aversion.

In order to sustain the above mentioned conclusion, historical data on assets returns available in USA (Shiller, 1989; Kockerlakota, 1996) shows that the average real return on large part of US stocks was ~7% per year, whilst the average real risk-free rate been ~1%.

The Effect of Background Risk

Beside the riskiness of assets returns there are also other sources of risk in determining the final wealth of an investor. For example the labour income (wages) are not fully risk-free, and for determine the effect of such risk we can introduce a zero-mean background risk $\tilde{\varepsilon}$ to initial wealth *w*. This leads to the adjustment of (2) resulting:

$$\alpha^{**} \in \arg\max_{\alpha} Eu\,(w + \tilde{\varepsilon} + \alpha\tilde{y}), \qquad (5)$$

Intuition might suggest that $\alpha^{**} < \alpha^*$, but we want to identify any kind of correlation between them which can influence the investors' decision making behaviour. The effect of $\tilde{\varepsilon}$ could be considered as "bad-luck" of an investor, therefor they would try to compensate the extra risk with a more precautious behaviour compared to $\tilde{y}$. We can rewrite the (5) considering this approach as:

$$\alpha^{**} \in \arg\max_{\alpha} Ev\,(w + \alpha\tilde{y}), \qquad (6)$$

where value of the function *v* is defined by $v(z) = Eu(w + \tilde{\varepsilon})$ for all *z*. In order to identify the difference we have to compare the function *u* with

v, in condition assumed that α^{**} (defined in (6)) is smaller than α^{*}(defined in

$$-\frac{v''(z)}{v'(z)} = -\frac{Eu''(z+\tilde{\varepsilon})}{Eu'(z+\tilde{\varepsilon})} \geq \frac{u''(z)}{u'(z)}\ , \tag{7}$$

for all $\tilde{\varepsilon}$ such that $E\tilde{\varepsilon} = 0$, which is equivalent to requiring $Eh(z, \tilde{\varepsilon}) \leq 0$, where

$$h(z,\epsilon) = u''(z+\epsilon)u'(z) - u''(z)u'(z+\epsilon).$$

Proposition 3.

(i) Any zero-mean background risk reduces the demand for other independent risks.

(ii) Absolute risk aversion is decreasing and convex.

We have to prove that (ii) is sufficient for condition (i). Noticing *A(.)* the absolute risk aversion, than we can write:

$$-Eu''(z+\tilde{\varepsilon}) = E[A(z+\tilde{\varepsilon})u'(z+\tilde{\varepsilon})].$$

Since we assumed that the absolute risk aversion is decreasing, results that the right hand side of this equation is larger than $EA(z+\tilde{\varepsilon})Eu'(z+\tilde{\varepsilon})]$. In addition, due to the fact that *A* is considered convex than results $EA(z+\tilde{\varepsilon})$ is larger than $A(z)$. These three observations together implies condition (7) which is necessary and sufficient for property (i).

Portfolio of Risky Assets

The above mentioned model can be further refined in sense that the risky assets could be split in two or more sub-portfolio, in this way the investors distribute the risk related they risky investments but also the earnings are distributed between the different portfolios. Let assume for simplicity that these two assets have the same distribution of returns $\tilde{x}_1$ and $\tilde{x}_2$. In order to determine the optimal structure of their portfolio, we should solve the following program:

$$\max_{\alpha} Eu(\alpha\tilde{x}_1 + (w-\alpha)\tilde{x}_2), \tag{8}$$

where α is the amount invested in the first risky asset. Since we consider the investor as risk-averse, the first-order condition is:

$$E(\tilde{x}_1 - \tilde{x}_2)u'(\alpha^*\tilde{x}_1 + (w-\alpha^*)\tilde{x}_2) = 0.$$

The unique rot of this equation is $\alpha^* = \frac{1}{2}w$, meaning that for risk-averse investors, the optimal investment is to perfectly balance their investment. This result, which is called also as risk-diversification, is reflected also by the fact that $\alpha^*\tilde{x}_1 + (w-\alpha^*)\tilde{x}_2$ is distributed as $\frac{\tilde{x}_1+\tilde{x}_2}{2} + \tilde{\varepsilon}$, where $\tilde{\varepsilon} \equiv (\alpha - \frac{1}{2})(\tilde{x}_1 - \tilde{x}_2)$

This means that the return of any portfolio α is distributed as return of the balanced portfolio plus a pure noise $\tilde{\varepsilon}$ which satisfy the condition:

$$E\left[\tilde{\varepsilon} \middle| \frac{\tilde{x}_1 + \tilde{x}_2}{2}\right] = 0.$$

This conducts us to the conclusion that accepting unbalanced portfolio is equivalent to accepting zero-means lotteries. Further, in real words investors tend to more diversify their portfolio, which is to dislike zero-mean risk.

As conclusion, based on theoretical approach, portfolio management is a simple problem, in which investors should not spend too much time and energy. Theoretically we assumed that financial markets are informationally efficient, meaning that the same information about the risk is available for all investors, and investors have mean-variance preferences. Investing more in risky assets investors expect higher return, and by diversifying the risky portfolio they looking for the optimum balance between assumed risk and expected returns.

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