Chapter 11

Econometric Models used in Macro-economic Analysis**

The economic situation in which correlations involves only two variables are very rare. Rather we have a situation where a dependent variable, Y, can depend on a whole series of variables factorial or regressor. In practice, there are correlations of the form:

$$Y = b_1 + b_2X_2 + b_3X_3 + \cdots + b_kX_n + \varepsilon$$

where values X_j $(j = 2, 3, ..., n)$ represents the variable factor or regressors, the values b_j $(j = 1, 2, 3, ...,k)$ are the regression parameters, and ε is the residual factor. Residual factor reflects the random nature of human response and any other factors, others than X_j, which might influence the variable Y.

We adopted the usual notation, respectively assigned to the first factor notation X_2, the second notation X_3 and so on. Sometimes it is convenient that the parameter b to be considered that coefficient of one variable X_1 whose value is always equal to unity. Then the relationship is rewritten as:

$$Y = b_1X_1 + b_2X_2 + b_3X_3 + \cdots + b_kX_n + \varepsilon$$

In the case of regression with two variables (E(ε) = 0), then, substituting, for given values of the variables X, we get:

$$E(Y) = b_1 + b_2X_2 + b_3X_3 + \cdots + b_kX_n$$

The relationship is multiple regression equation. For now, conventional, we consider that it is the linear form. Unlike the case of two-variable regression, we cannot represent this equation in a two-dimensional diagram. b_j are regression parameters. Sometimes, they are also called regression coefficients. b_1 is a constant (intercept) and b_2, b_3 and so on, are the regression slope parameters. b_4 measuring the effects of E(Y) produced by changing one unit of X_4,considering that

** This chapter is based on some elements included in the article *Macro-economic Analysis based on Econometric Models*, authors prof. Constantin Anghelache PhD, prof. Gabriela Victoria Anghelache PhD, prof. Ioan Partachi PhD, Emilia Stanciu PhD student, Bogdan Dragomir PhD student, RRS Supplement nr. 10/2014 all other factor variables remain constant. b_2 measures the effects on $E(Y)$ produced by changing one unit of X_2, considering that all other variables remain constant factor. As the population regression equation is unknown, it has to be estimated based on data sample. Suppose that we have available a sample of n observations, each observation containing the dependent variable values for both Y and for each factorial variables X. We write the values for observation i as: Y_i , X_{2i} , X_{3i} , X_{4i} ,..., X_{ki}

For example, X_{26} is the value of X_2 in the 6th observation and X_{35} is the value X_3 taken in the 5th observation. For a similar manner, Y_8 is the variable Y in the observation of 8 and so on[1].

Given that it is assumed that the sample data were generated by the correlation of the population, each observation have to involve a set of values satisfy the multiple equation regression.

We can write the equation: $Y_i = b_1 + b_2X_{2i} + b_3X_{3i} + ...+ b_kX_{ki} + \varepsilon_i$ for all the values, where ε_i represents the residual value for the observation of the i.

We can rewrite the relationship in a simple matrix form, as follows:
$Y = X\beta + \varepsilon$, where X is a matrix the form of n x k with k column of values and then all sample values of the $k - 1$, X variables.

Thus, the fourth column of X, for example, contains the values of X_4 of the sample n, the seventh column contains the values of X_7 and so on. b is a vector of k x 1 column containing the parameters b_j and ε is an vector of n x 1 column containing the residual values.

The effective values of Y will not coincide with the expected values of Y and, in the case of two-variable regression, the differences between them are known as residual values.

Like $Y_i = \hat{Y}_i + e_i$, for all values of i where e_i is the residual corresponding to the observations of i. The relationship can be written

[1] Anghelache, C., Negoiţă, I. et. al. (2013) - „*Inflation and Unemployment – a Correlative Analysis*", International Symposium "Romania and the Economic-Financial Crisis. Methods and Models for Macroeconomic Analysis", May 2013, "Artifex" University of Bucharest, published in Romanian Statistical Review, Supplement no. 2/2013, pp.22 – 29

as: $Y_i = \hat{b}_1 + \hat{b}_2 X_{2i} + \hat{b}_3 X_{3i} + \hat{b}_{ki} X_{ki} + \cdots + e_i$, for all values of i or on matrix form: $Y = X\hat{b} + e$, where X and Y are already defined.

There are two issues to be retained on the residual values.

First, regardless of the method used to estimate the regression equation, we get such residual values - one for each of the sample observations. Second, as expected $\hat{b}_j$ when it becomes known and can be used to calculate them. Now, we need to calculate the differential with the vector $\hat{b}$ and equalizer to zero the result. Such of this matrix lead to the following relation:

$$\frac{\partial S}{\partial \hat{b}} = -2X'Y + 2X'X\hat{b} = 0$$

The above equation is a set of k equations that can be written as $X'X\hat{b} = X'Y$.

- **The Linear and Non-displaced Estimator**

A general matrix demonstration on the features BLUE in case of multiple regression is outside the goal.

We will limit ourselves only to find expressions for the variations and covariance of OLS estimators.

As we shall see, these expressions are important if we want to develop inferences about the parameters of the multiple regression.

The matrix is known as the matrix variation – covariation of the vector $\hat{b}$, which generally is written in the form of $VAR(\hat{b})$. Note that in the bottom of its main diagonal, it contains variations vector $\hat{b}_j$. Outside the diagonal elements represents the covariance between different values $\hat{b}_j$that would result in case of more samples extraction. It is clear that if we need to develop inferences about the true value of $b_{j,}$ it is necessary to find an expression for this matrix. The equation is just the expression for the matrix variation – co-variation of vector OLS $\hat{b}$.

We write the element of row i and column j of the $(X'X)^{-1}$ inverse matrix as X^{ij}. Since $(X'X)^{-1}$ is symmetric, we have $X^{ji} = X^{ij}$. The comparison indicates that the variation of $\hat{b}_j$, which is written in the form $\sigma^2_{\hat{b}_j}$ is given by: $\sigma^2_{\hat{b}_j} = VAR(\hat{b}_j) = \sigma^2 X^{ij}, j = \overline{1,k}.$

Thus, to find the variation of $\hat{b}_j$, we have to take the j element of the diagonal matrix $(X'X)^{-1}$ and to multiply it by the change in average residual values, σ^2. Square root of $VAR(\hat{b}_j)$ is known as the standard error of $\hat{b}_j$ and it is noted by $\sigma_{\hat{b}_j}$.Comparing the two equations further, resulting that:

$cov(\hat{b}_i, \hat{b}_j) = \sigma^2 X^{ij}$ for all values $i \neq j$

The expressions obtained are of considerable importance for inference in multiple regression.

It is possible to obtain equivalent expressions if we work in terms of deviations of variables from their average. It is merely necessary to work in terms of the inverted matrix $(x'x)^{-1}$ instead of the matrix $(X'X)^{-1}$. A complete derivation would prove repetitive but it is not difficult to prove that: $\sigma^2_{\hat{b}_j} = VAR(\hat{b}_j) = \sigma^2 x^{ij}, j = \overline{1,k}$ and $cov(\hat{b}_i, \hat{b}_j) = \sigma^2 x^{ij}$ for all values $i \neq j$ where x^{ij} is the element from row $(i-1)$ and column $(j-1)$ of matrix $(x'x)^{-1}$.

It should be noted that the relationship does not lead to an expression for $VAR(\hat{b}_j)$.

In the particular case of two-variable regression, $(\mathbf{x'x})^{-1}$ is only the scalar $\sum x_2^2$ so that $x^{22} = 1/\sum x_2^2$, which leads to the relationship:

$VAR(\hat{b}_2) = \sigma^2/\sum x_2^2$

This is identical to the corresponding expression for the variance estimator OLS for the slope regression parameter with two variables.

- **Additional properties of the model**

As in the case of two-variable regression, where the OLS estimators have to to be mainly but not only stationary and asymptotically efficient and effective, it is necessary that the IID assumption of classic model to self-sustain – that means the residual values have to be normally distributed[2]. Therefore, if OLS estimators

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[2] Anghelache, C., Negoiţă, I. et. al. (2013) - *„Inflation and Unemployment – a Correlative Analysis*", International Symposium "Romania and the Economic-Financial Crisis. Methods and Models for Macroeconomic Analysis", May 2013, "Artifex" University of Bucharest, published in Romanian Statistical Review, Supplement no. 2/2013, pp.22 – 29

must have these properties, it is necessary that all classical assumptions are valid. A proof of ownership efficiency is outside the goal we have set. Remember only that efficiency implies that OLS estimators have the minimum variance of stationary estimators of all - not only of linear stationary estimators.

Normality residual values have other two important consequences for OLS regression. First, it means that the distributions of OLS estimators will be the selection of normal distributions. A demonstration of this statement is analogous to the case of two-variable regression. However, that, whereas under all classical assumptions, each $\hat{b}_j$ is stationary with set variation: $\hat{b}_j$ is $N\left(b_j, \sigma^2 X^{ij}\right), j = \overline{1,k}$.

A precise knowledge of the distributions of selection of OLS estimators, respectively $\hat{b}_j$, is of vital importance for inference. It is often helpful for the relationship to be expressed in an alternative form, working in terms of deviations from their average variables X, in which case we get: $\hat{b}_j$ is $N\left(b_j, \sigma^2 x^{ij}\right), j = \overline{1,k}$.

The second consequence of the assumption of normality of the distribution of residual factors is, as in the case of two-variable regression, that OLS estimators are maximum probability estimator. As in the two-variable regression, MLE for σ^2 's proves:

$$\tilde{\sigma}^2 = \frac{\sum e_i^2}{n}$$

where $\sum e_i^2$ is the sum of squared residual factors. However, $\tilde{\sigma}^2$ proves to be an moved estimator of σ^2 real. In fact, it can be shown that if multiple regression:

$$E(\tilde{\sigma}^2) = \frac{n-k}{n} \cdot \sigma^2 \neq \sigma^2$$

and is a generalization of the two-variable regression results.

Since, under classical assumptions, OLS and ML estimators of the parameters b_j are identical at this point it may seem that ML estimation contribute little to our analysis of regression equations.
Maximum probability estimation becomes most relevant when classical hypotheses are refused.

For example, this method is often used in cases where the regression equation is *nonlinear*.

There is also of great importance when classical assumptions on the variable factor and/or to the residuals are invalidated. As we have seen, if the classical assumptions are not valid, then the OLS estimators lose some, or all, of the desired properties. It proves that, in such circumstances, OLS estimators and ML estimators *are not identical*. In such a situation, ML estimators have the advantage that they still maintain their properties, namely compatibility and asymptotic efficiency.

- **Inference in multiple regression**

By condition that all classical assumptions are valid, inferences on the slope parameters in multiple regression can be based on the outcome $\hat{b}_j = N\left(b_j, \sigma^2 x^{ij}\right)$ which implies that, for $j = \overline{1, k}$ have:

$$\frac{\hat{b}_j - b_j}{\sigma_{\hat{b}_j}}$$

has a distribution N(0, 1).

We will focus on the slope parameters that are of interest. The inference on the parameter b_1, it should be based on the set equation, with $j = 1$.

The problem that arises it is that the standard errors, $\sigma_{\hat{b}_j}$, are unknown because the residuals variations, σ^2 is unknown.

When we substituting $\sigma_{\hat{b}_j}$ on stationary estimators, $s_{\hat{b}_j}$, as in the two-variable regression, we have to change the distribution of *t*. It can be shown that:

$$\frac{\hat{b}_j - b_j}{s_{\hat{b}_j}}$$

has a distribution *t*, cu *n – k g.l.*

For example, at a 95% confidence interval for any value β_j (j = 2, 3, ...,k) is: $\hat{b}_j + t_{0,025} s_{\hat{b}_j}$

the value of $t_{0,025}$ depending on *n – k* and on the number of degrees of freedom. In order, to obtain a 99% interval are replaced $t_{0,025}$ with $t_{0,005}$.

Trustworthiness check can continue over the similar lines of the two-variable regression determined[3].

To test the null hypothesis like H_0 : $b_j = 0$ ($j = 2, 3, ..., k$), we have to say that under the null hypothesis, which implies: $\hat{b}_j / s_{\hat{b}_j}$ have a *t* distribution with *n - k degrees of freedom.*

Therefore, we use $\hat{b}_j / s_{\hat{b}_j}$ like test statistic and reject the null hypothesis that variable X_j does not influence the variable *Y* whether the absolute value of the test statistic is sufficiently large.

As in the case of two-variable regression, the statistical test is often called the coefficient *t.*

[3] Anghelache, C., Voineagu, V., Negoiţă, I. et. al., „*The Features of the Chronological Series of Statistical Indices*", International Symposium "Romania and the Economic-Financial Crisis. Methods and Models for Macroeconomic Analysis", May 2013, "Artifex" University of Bucharest, published in Romanian Statistical Review, Supplement no. 2/2013, pp. 55-61