The Markowitz Model

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Abstract

Harry Max Markowitz, laureate of the Nobel Prize for economy in 1990, is known for is contributions to the development of the modern theory of the portfolio, his studies being focused with priority to the effects which the risk, the profitableness or the assets correlation are implying when the forming of a portfolio is aimed. By consecrating to the study of this matter, he contributed fundamentally to the settlement of the financial markets issues as from the years 1950, by elaborating a modern theory concerning the efficiency of the choice within a portfolio. The theory allows the setting up of an optimum modality to place the capitals as well as the diminishing of the risk degree for the done financial investments.

Key words: *portfolio, risk, capital, investment, financial market*

Markowitz has study profoundly the motivation of the behaviour of the players on the financial market from the point of view of the tendencies to maximize the profitableness and to decrease the risk.

The Markowitz theory generated an economic-mathematical model aiming the behaviour of the subjects of the financial market. He introduced the practice of the diversification of the assets portfolios depending on the correlation between risk and profitableness.

In the frame of the model he conceived, the assets are correlated two by two and it is possible to identify the proportion of the assets within the portfolio in order to identify **the portfolio with absolute minimum variant.** The Markowitz model starts from the idea that to any **risk can be**

associated with a probability of occurrence in the asset, being the more so risky so as there is a higher volatility of the gains.

In the present context of the economic development, the Markowitz model keeps its relevance and keeps on being utilized for identifying the optimum choices in the frame of the portfolio. The investor cannot afford the luxury to place his entire available capital in a single asset or in a single business[1]. By his model, Markowitz offered a basis for the analysis of the portfolio of financial assets and for setting up the optimum from the financial point of view taking into consideration the evolutions of the individual yields of the assets and the risk associated to them.

The first step towards building up a portfolio consists of building it up with two assets. In these conditions, the hypotheses of the model being proposed by Markowitz are the following:

- the investors consider each investment alternative as being represented by the distribution of the expected profit probabilities, over a period of time;
- the investors maximize the expected utility over a period of time while the curve of the utility maximizes the marginal utility of their welfare;
- the investors estimate the risk on the basis of the alteration in the expected profits;
- the investors take decisions on the basis of the expected risk and profit only, so that the utility curve is expressed as a function of the expected profit and profit variant;
- for a given level of the risk, the investors prefer a high profit; for a given level of the expected profit, the investors prefer the smaller risk[2].

Let's assume that on a stock exchange market there are three assets transacted, noted with i, $i = \overline{1,3}$. For the three assets with risk, the following data are known[3]:

- The expected profitableness:
  $E(R_i)$: $E(R_1) = 15\%$, $E(R_2) = 18\%$
- The individual risks:

[1] Here we can apply the well-known dictum „We must not out the eggs in a single panel'

[2] " I want to point out that, as to the theoretical part, it would be necessary to show which are the formulas characterising the Markowitz model, but when solving the problem I shall write all of them in order to allow an as good as possible understanding on the mode to solve it

[3] The equations were represented in a third-part software

$\sigma_i: \sigma_1 = 20\%, \sigma_2 = 22\%$ and $\sigma_3 = 26\%$

- The correlation coefficients between the three assets:

  $\rho_{12} = -0.15, \rho_{13} = -0.4$

  $\rho_{23} = -0.10$

  Considering the above:

A. We consider a rational investor with behaviour of Markowitz type: this one aims to get a given yield with a minimum risk. Knowing that the profitableness which this one wishes to obtain is p = 20%, it is required for this case to establish the structure of the portfolio he has to form and the risk associated to the portfolio.

B. The setting up of the Markowitz border equation (the upper part) and the equation of the tangent to this border in the point established by the requirement from A.

Solution: first of all, based on the data out of the problem the matrices[4] have to be built up:

$$\Omega = \begin{pmatrix} 0.0400 & -0.0066 & 0.0208 \\ -0.0066 & 0.0484 & -0.0057 \\ 0.0208 & -0.0057 & 0.0676 \end{pmatrix}$$

$$\Omega^{-1} = \begin{pmatrix} 30.2013 & 3.0506 & -9.0346 \\ 3.0506 & 21.1780 & 0.8533 \\ -9.0346 & 0.8533 & 17.6450 \end{pmatrix}$$

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} E(R_1) \\ E(R_2) \\ E(R_3) \end{pmatrix} = \begin{pmatrix} 0.15 \\ 0.18 \\ 0.23 \end{pmatrix}$$

- **Solving the point A:**

[4] Calculations were made in Excel

$$\min \frac{1}{2}\sigma_p^2 = \min \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} X_k X_j \sigma_{kj} = \min \frac{1}{2} X^T \Omega x$$

having restrictions : $\begin{cases} \sum_{k=1}^{n} X_k \mu_k = \rho \quad X^T \mu \quad \text{p is fixed} \\ \sum_{k=1}^{n} X_k = 1 \quad X^T e = 1 \end{cases}$

By solving this problem, the outcome shows that:

- The optimum structure of the investor's portfolio:

$$X^* = \lambda_1 \Omega^{-1}\mu + \lambda_2 \Omega^{-1} e \quad \text{sau} \quad X^* = \frac{1}{D}\left[(A\rho - B)\Omega^{-1}\mu + (C - \rho B)\Omega^{-1} e\right]$$

- The risk of the investor's portfolio:

$$\sigma_p^2 = X^{*T}\Omega X^*$$

or

$$\sigma_{p^*}^2 = \frac{1}{D}\left[A\rho^2 - 2B\rho + C\right]$$

where the following notation applies[5]:

$$e = (1,1,\ldots,1)^T, \quad \lambda_1 = \frac{A\rho - B}{AC - B^2}, \quad \lambda_2 = \frac{C - \rho B}{AC - B^2}$$

$$A = e^T \Omega^{-1} e, \quad B = e^T \Omega^{-1} \mu, \quad C = \mu^T \Omega^{-1} \mu$$

[5] We have used the notations from the course of mister professor Moisă Altar, the mode of solving the problem being the suggested one through the course support.

And

$$D = AC - B^2$$

By making the calculations, we have:

$$\Omega^{-1}e = \begin{pmatrix} 30.2013 & 3.0605 & -9.0346 \\ 3.0506 & 21.1780 & 0.8533 \\ -9.0346 & 0.8533 & 17.6450 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 24.2174 \\ 25.0820 \\ 9.4637 \end{pmatrix}$$

$$\Omega^{-1}\mu = \begin{pmatrix} 30.2013 & 3.0605 & -9.0346 \\ 3.0506 & 21.1780 & 0.8533 \\ -9.0346 & 0.8533 & 17.6450 \end{pmatrix} \bullet \begin{pmatrix} 0.15 \\ 0.18 \\ 0.23 \end{pmatrix} = \begin{pmatrix} 3.0014 \\ 4.4659 \\ 2.8568 \end{pmatrix}$$

$$A = e^T\Omega^{-1}e = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \bullet \begin{pmatrix} 24.2174 \\ 25.0820 \\ 9.4637 \end{pmatrix} = 58.7631$$

$$B = e^T\Omega\mu = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \bullet \begin{pmatrix} 3.0014 \\ 4.4659 \\ 2.8568 \end{pmatrix} = 10.3240$$

$$C = \mu^T\Omega^{-1}\mu = \begin{pmatrix} 0.15 & 0.18 & 0.23 \end{pmatrix} \bullet \begin{pmatrix} 3.0014 \\ 4.4659 \\ 2.8568 \end{pmatrix} = 1.9111$$

$$\lambda_1 = \frac{A\rho - B}{D} = 0.2498$$

$$\lambda_2 = \frac{C - \rho B}{D} = -0.0269$$

$$X^{*}=\lambda_1\Omega^{-1}\mu+\lambda_2\Omega^{-1}e=0.2498\bullet\begin{pmatrix}3.0014\\4.4659\\2.8568\end{pmatrix}+(-0.0269)\bullet\begin{pmatrix}24.2174\\25.0820\\9.4637\end{pmatrix}=\begin{pmatrix}0.0990\\0.4416\\0.4594\end{pmatrix}$$

Therefore: $X_1^* = 9.90\%$, $X_1^* = 44.16\%$, $X_1^* = 45.94\%$, therefore it is verified that:

$\rho = X^{*T}\mu = 0.0990*0.15 + 0.4416*0.18 + 0.4954*0.23=0.20=20\%$.

$$\sigma_p^2=X^{*T}\Omega X^{*}=(0.0990\quad 0.4416\quad 0.4594)\bullet\begin{pmatrix}0.0400 & -0.0066 & 0.0208\\-0.0066 & 0.0484 & -0.0057\\0.0208 & -0.0057 & 0.0676\end{pmatrix}\begin{pmatrix}0.0990\\0.4416\\0.4594\end{pmatrix}=0.0231$$

Or

$$\sigma_{p^*}^2=\frac{1}{D}\left[A\rho^2-2B\rho+C\right]=\frac{1}{5.7180}\left(58.7631*0.20^2-2*5.7180*0.20+1.9111\right)=0.0231$$

Wherefror the risk of portfolio results:

$$\sigma_p=\sqrt{\sigma_p^2}=\sqrt{0.0231}=0.1520=15.20\%$$

- **Solving point B:**

$$R_p=\frac{B}{A}+\sqrt{\frac{D}{A}}\bullet\sqrt{\sigma_p^2-\frac{1}{A}}$$

$$R_p=\frac{10.3240}{58.7631}+\sqrt{\frac{5.7180}{58.7631}}\bullet\sqrt{\sigma_p^2-\frac{1}{58.7631}}$$

$$R_p=0.1757+0.3119\bullet\sqrt{\sigma_p^2-0.0170}$$

With

$$\sigma_p\geq\sqrt{\frac{1}{A}}\approx\sqrt{0.0170}=0.1305$$

The equation of a line is: $y - y_0 = m(x - x_0)$, where m is the slope of the line. In a portfolio the line is equal to the derivate. We shall have:

$$R-R_p=\sqrt{\frac{D}{A}}\bullet\frac{\sigma_p}{\sqrt{\sigma_p^2-\frac{1}{A}}}(\sigma-\sigma_p)$$

Where the slope is:

$$m=\frac{dR_p}{d\sigma_p}=\sqrt{\frac{D}{A}}\bullet\frac{\sigma_p}{\sqrt{\sigma_p^2-\frac{1}{A}}}$$

The tangent equation in the point P(0.1520 ; 0.20) is:

$$R-0.20=0.3119\bullet\frac{0.1520}{\sqrt{(0.1520)^2-\frac{1}{58.7631}}}(\sigma-0.1520)$$

The intersection with the ordinates axis is obtained for $\sigma = 0$ and is given by:

$$R=R_p-\sqrt{\frac{D}{A}}\bullet\frac{\sigma_p^2}{\sqrt{\sigma_p^2-\frac{1}{A}}}=\frac{B}{A}-\frac{1}{A}\sqrt{\frac{D}{A}}\bullet\frac{1}{\sqrt{\sigma_p^2-\frac{1}{A}}}$$

Meaning R= 0.1076=10.76%.

Conclusions

In the present context of the economic development, the Markowitz model keeps its relevance and keeps on being utilized for identifying the optimum choices in the frame of the portfolio.

Through the practical application submitted and solved above, I aimed to show how a portfolio can be created by using the model proposed by Markowitz. Thus, for the three financial assets for which both the profitableness and the risk for each of them are known, I have built up a portfolio with a fixed yield of 20%, under minimum risk conditions.

We have found out the portfolio structure as being: X_1^* = 9.90%, X_1^* = 44.16%, X_1^* = 45.94%, while the minimum risk of the portfolio is 15.20%. By correlating the results with the initial data of the problem we can notice that an investor prefers to invest the less in the asset Xi, which shows the lowest yield and risk and prefers to invest an appropriate proportion in the assets of the highest risk, X2 and X3, but with a more attractive profitableness. By building up the portfolio, the investor bearing a behavior of Markowitz type succeeded to get a risk of 15.2%, a value below the risk of any other asset of the portfolio, here the significance of the Markowitz model being observed to the best.

From my point of view, the submitted application showed the mode of building up a portfolio according to the Markowitz model, emphasizing its utility and succeeding, meantime, to build up the efficient border as well, on the basis of a number of numerical date.

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