FROM THE INDEX NUMBERS’ METHOD TO THE METHOD OF COEFFICIENT OF ELASTICITY

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Abstract

Using the simple method of index numbers, and synthesizing the originality of his excellent statistical thinking by definition, this article identifies and presents an inimitable shortcut from Index – Numbers’ method to elasticity method. A final remark underlines the beauty and the rigour of this scientific demarche specific for the statistical thinking. This paper is a real homage addressed to Professor M. C. Demetrescu, and to his remarkable PhD thesis, printed approximately half a century ago, one of the best statistic and economic book about population demand1.

Keywords: statistical thinking, index or index – number, coefficient of elasticity.

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The relationship between the Index – Number and the coefficient of elasticity identifies the essence of represented phenomena, respectively elasticity of demand or of supply and represents a statistical correlation between two pure methods, using the inimitable interrogation and demonstration cycles of statistical thinking, synthesized in the outstanding doctoral thesis of our regretted Professor M.C. Demetrescu, the Romanian marketing advocate with statistical arguments, for more than five decades.

If Index-Number is used to measure the change of a quantitative nature, which cannot be directly observed (Bowley, 1920) or shows either the increase, or the decrease in a value that is hardly likely to enjoy an (extremely) accurate measurement (Edgeworth, 1925), the Index-Numbers’ method is an unique statistical method, unused by other science directly. This means that this method represents “pure” statistics and nothing else more...

Another proof of the same idea of index statistical “purity” is Henri Guitton’s definition: “whenever a variable modifies its level, in time or space, a statistical index is generated…” The general method of decomposition and

1. “Elasticitatea cererii populației cu privire la bunurile de consum și servicii” - Demetrescu M.C. / 1967, Editura Academiei, Bucharest
factorial analysis, based on Index – Number and used for understanding the practical phenomena as a system became soon the Index – Numbers’ method. The evolution of this method after more than three centuries had solved various theoretical and methodological problems concerning the calculation method used, including the formula, choosing the base and the system of weighting / balancing, and, more especially, the practice way of construing, during a long process, from the first constructions of Dutot or Carli

\[ \text{Dutot} = \frac{\sum_{i=1}^{n} p_i}{\sum_{i=1}^{n} P_i}, \quad \text{Carli} = \frac{1}{n} \sum_{i=1}^{n} \frac{p_i}{P_i}, \]

where: \( p_i \) and \( P_i \) are the prices of the current, and base periods, to the Laspeyres or Paasche classical indices:

\[ \text{Laspeyres} = \frac{\sum_{i=1}^{n} p_i q_{i0}}{\sum_{i=0}^{n} p_i q_{i0}}, \quad \text{and} \quad \text{Paasche} = \frac{\sum_{i=1}^{n} p_i q_{i1}}{\sum_{i=0}^{n} p_i q_{i1}} \]

where \( p_{i0} \) and \( p_{i1} \) are prices of the base period (0), and current prices (1) and \( q_{i0} \) and \( q_{i1} \) = quantities of the base period (0), and current quantities (1). The processes of generating new indices or of the multiplication of the calculus formulae have revealed two tendencies: extreme axiomatization and mathematization, following the patterns of Tornqvist and Divisia indices (e.g. \( \ln (\text{Tornqvist Index}) = \sum \frac{1}{2} \left[ \frac{p_i q_{i1}}{\sum p_i q_{i1}} + \frac{p_i Q_{i1}}{\sum p_i Q_{i1}} \right] \times \ln \frac{p_i}{P_i} \) ) and culminating in the school of axiomatic indices; variants of integrating the additive, or the mixed additive–multiplicative models of construction, which come closer to the signification of real phenomena as logical vein of economic signification of the index constructions, specific to the latest construction worldwide at the end of the 20th century (e.g. David Neven Index)

The difference between Paasche \( \left( \frac{P_{i1}}{P_{i0}} \right) \) and Laspeyres \( \left( \frac{L_{i1}}{L_{i0}} \right) \) classical indices presents a specific importance for statistics. A lot of statisticians have written many pages about this difference, beginning with L. Bortkiewicz, and just apparently finishing with D Neven, in international statistics’ world, and
similar to our national statistics from M.C. Demetrescu to L. Tővissi. Using the simple method of Index – Numbers in his doctoral thesis, M.C. Demetrescu had identified an incredible shortcut from Index – Numbers’ method to elasticity method, in 1967, and this paper offers a succinct demonstration of this remarkable demarche.

If the above difference is compared as a ratio to the quantity-weighted price index of the base period \( \left( \frac{L^p_{1/0}}{L^p_{0/1}} \right) \), then:

\[
\frac{p^p_{1/0} - L^p_{1/0}}{L^p_{1/0}} = \left( \frac{\sum_i p_{1i}q_{1i}}{\sum_i p_{0i}q_{1i}} \right) \cdot \left( \frac{\sum_i p_{1i}q_{0i}}{\sum_i p_{0i}q_{0i}} \right)
\]

One can process the difference aside from the substitutions:

\[
p_{1i}q_{1i} = p_{1i} \cdot \frac{p_{0i}}{p_{0q_{1i}}} \cdot q_{1i} = p_{1i} \cdot q_{1i} \cdot \frac{p_{1i}q_{1i}}{p_{0q_{1i}}}
\]

and

\[
p_{0i}q_{1i} = \frac{p_{0i}q_{1i}}{p_{0q_{1i}}} \cdot p_{0i}q_{1i}
\]

and thus to obtain:

\[
p^p_{1/0} = \frac{\sum_i p_{1i}q_{1i} \cdot p_{1i/0} \cdot q_{1i/0}}{\sum_i p_{0i}q_{1i}} \times \frac{\sum_i p_{0i} \cdot q_{1i} \cdot q_{1i/0}}{\sum_i p_{0i}q_{1i}}
\]

Thus stated the difference can be written in the form:

\[
\frac{p^p_{1/0} - L^p_{1/0}}{L^p_{1/0}} = \left[ \frac{1}{\sum_i p_{0i}q_{1i}} \cdot \sum_i p_{0i} \cdot q_{1i} \cdot p_{1i/0} \cdot q_{1i/0} - \sum_i p_{0i} \cdot q_{1i} \cdot p_{1i/0} \cdot q_{1i/0} \right] \cdot \left[ \frac{\sum_i p_{1i}q_{1i}}{\sum_i p_{0i}q_{1i}} \right]
\]

And after the processing is:

\[
\frac{1}{\sum_i p_{1i}q_{1i}} \cdot \sum_i p_{1i}q_{1i} \cdot p_{1i/0} \cdot q_{1i/0} - \sum_i p_{1i}q_{1i} \cdot p_{1i}q_{1i}
\]
However: \[ \sum p_i q_{i0} \] as can also be written

\[ \frac{\sum p_i q_{i0} p_{i1/0}}{\sum p_i q_{i0}}, \] and \[ \frac{\sum p_i q_{i1}}{\sum p_i q_{i0}} \] by analogy, is also \[ \frac{\sum p_i q_{i0} q_{i1/0}}{\sum p_i q_{i0}} \] and the result of replacing it in the formula is:

\[ \frac{P^p_{1/0} - L^p_{1/0}}{L^p_{1/0}} = \frac{1}{\sum p_i q_{i0}} \cdot \frac{1}{\sum p_i q_{i0}} \left[ \frac{\sum p_i q_{i0} p_{i1/0} q_{i1/0} - \sum P_{i0} q_{i0} q_{i1/0}}{\sum p_i q_{i0} q_{i0}} \right] \]

But as the covariance between X and Y is equal to \[ \sum \frac{X}{N} - \frac{1}{X Y} \] and \( X = p_{i1/0} \) and \( Y = q_{i1/0} \) weighted covariance formula of the previous relationship is found in the bracket. Starting from \( \text{cov} \ (p_{i1/0} \ q_{i1/0}) = \omega(r_{p_{i1/0} \ q_{i1/0}}) \cdot \omega(\sigma_{p_{i1/0}}) \cdot \omega(\sigma_{q_{i1/0}}) \) the result is:

\[ \frac{P^p_{1/0} - L^p_{1/0}}{L^p_{1/0}} = \omega(r_{p_{i1/0} \ q_{i1/0}}) \cdot \frac{\omega(\sigma_{p_{i1/0}})}{\sum p_i q_{i0}} \cdot \frac{\omega(\sigma_{q_{i1/0}})}{\sum p_i q_{i0}} = A \cdot B \cdot C = \Delta(\%) \]

The correlation coefficient \( r_{(x, y)} = \frac{\text{cov}(x, y)}{\sigma_X \cdot \sigma_Y} \) is

\[ r_{(p_{i1/0} \ q_{i1/0})} = \frac{\text{cov}(p_{i1/0} \ q_{i1/0})}{\sigma_{p_{i1/0}} \cdot \sigma_{q_{i1/0}}}, \] and where \( w = q_{i0} p_{i0} \) the result is:

\[ \text{cov}(p_{i1/0} q_{i1/0}) = \omega(r_{p_{i1/0} q_{i1/0}}) \cdot \omega(\sigma_{p_{i1/0}}) \cdot \omega(\sigma_{q_{i1/0}}) \]

The relative difference between the two interpreter indices Paasche \( (p^p_{1/0}) \) and Laspeyres \( (L^p_{1/0}) \) type is equal to the product of:

• weighted correlation coefficient between individual price indices for \( p_{i1/0} \) and the quantities for \( q_{i1/0} \) (A),
• weighted coefficient of variation for \( p_{1/0} \) (B),
• weighted coefficient of variation of \( q_{1/0} \) (C).

The relative difference between the two indices is expressed and interpreted differently weighted by the regression coefficient based on the formula:

\[
b_{p_{1/0}q_{1/0}} = \frac{\text{cov}(p_{1/0}q_{1/0})}{\sigma_{p_{1/0}}^2}\]

that after calculation becomes:

\[
\frac{\omega(r_{p_{1/0}q_{1/0}}) \cdot \omega(\sigma_{p_{1/0}})}{\sum p_{i0}q_{i}\sum p_{i0}q_{i}} \cdot \frac{\omega(\sigma_{q_{1/0}})}{\sum p_{i0}q_{i}} = \frac{\omega(\text{cov}(p_{1/0}q_{1/0})) \cdot \omega(\sigma_{p_{1/0}}) \cdot \omega(\sigma_{q_{1/0}})}{\sum p_{i0}q_{i} \cdot \sum p_{i0}q_{i}}
\]

\[
= \frac{\omega(\text{cov}(p_{1/0}q_{1/0})) \cdot \omega(\sigma_{p_{1/0}})}{\sum p_{i0}q_{i} \cdot \sum p_{i0}q_{i}} \cdot \frac{1}{\sum p_{i0}q_{i} \cdot \sum p_{i0}q_{i}}
\]

Multiplying the numerator and denominator \( \omega(\sigma_{p_{1/0}}) \sum p_{i0}q_{i} \) and get:

\[
= \frac{\omega(\text{cov}(p_{1/0}q_{1/0})) \cdot \omega(\sigma_{p_{1/0}}) \cdot \sum p_{i0}q_{i}}{\sum p_{i0}q_{i} \cdot \sum p_{i0}q_{i}} = \frac{1}{\sum p_{i0}q_{i} \cdot \sum p_{i0}q_{i}} = A \cdot B \cdot C = \Delta(\%)
\]
This second equation relative difference between the two interpreter indices Paasche \( P_{0/1}^P \) and Laspeyres \( P_{0/1}^L \) type is equal to the product of:

- weighted regression coefficient (A) expressing the regression line slope value of \( q_i \) and \( p_i \) and sensitivity relationship (\( A' \)),
- squared coefficient of variation for \( p_i 1/0 \) (\( B' \)),
- and average \( p_i 1/0 \) to \( q_i 1/0 \) (\( C' \)).

The relative size of the difference results from the association existing between price changes and changes in the sold quantities. Based on the regression coefficient \( b \) is reached coefficient of elasticity (\( \lambda \)):

\[
\lambda_{c/p} = \frac{\Delta c}{c} \left( - \frac{\Delta p}{p} \right) \quad \text{and} \quad \lambda_{q/p} = \frac{\Delta q}{q} \left( - \frac{\Delta p}{p} \right)
\]

But \( \frac{\Delta c}{\Delta p} \) or \( \frac{\Delta q}{\Delta p} \) represent the regression coefficient \( b \) of \( q \) by \( p \).

Hence:

\[
\lambda_{c/p} = - \frac{p}{c} b \quad \text{or} \quad \lambda_{q/p} = - \frac{p}{q} b
\]

(where \( p \) and \( q \) are the corresponding observed values, or calculated by the equation \( q = a + bp \))

The calculations generally use the average value of \( p \) and \( q \) and \( c \)

\[
(\lambda_{q/p} = - \frac{p}{q} b) \quad \text{bsau} \quad \lambda_{c/p} = - \frac{p}{c} b).
\]

The formula previously used generates:

\[
\lambda_{q/p} = - \frac{\omega}{\omega(\sigma^2 p_{1i/0})} \sum p_{i1}q_{i0} - \sum p_{i0}q_{i1} \quad \sum q_{i1}p_{i0} \quad \sum q_{i0}p_{i1}
\]

and the coefficient of elasticity expressed as a percentage the share that \( q_{i1/0} \) varies when \( p_{i1/0} \) changes by 1%.
By means of Laspeyres interpreter index, the coefficient of elasticity is shown in the following formula:

$$\lambda_{q/p} = - \frac{\omega(\text{cov}(p_{1/0}, q_{1/0}))}{\omega(\sigma^2_p_{1/0})} \frac{L^p_{1/0}}{L^q_{1/0}}$$

Another solution to obtain the coefficient of elasticity is based on the application of regression analysis to the method indicated in directly. Assuming a “perfect” correlation between q and p, and processing the regression equation $Y = a + bx$, where $a = \overline{Y} - b\overline{x}$ introduced in $Y = a + bx$ is $Y = \overline{Y} - b\overline{x} + b$.

From this is obtained $b = \frac{Y - \overline{Y}}{X - \overline{X}}$ respectively $b = \frac{q_{a_{1/0}} - a_{1/0}}{p_{a_{1/0}} - \overline{p}_{1/0}}$

or in the alternative of the weighted regression coefficients:

$$\omega(b_{q/p}) = \frac{\sum q_{a_{1/0}}p_{0}}{\sum p_{0}} - \frac{\sum q_{0}p_{1/0}}{\sum p_{1/0}q_{1/0}}$$

Substituting this value with the known difference from now on:

$$\frac{P^p_{1/0} - L^p_{1/0}}{L^p_{1/0}} = \left[ a_{1/0} - \frac{\sum q_{a_{1/0}}p_{0}}{\sum q_{0}p_{1/0}} \right] \left[ \frac{\sum q_{0}p_{1/0}}{\sum p_{1/0}q_{1/0}} \right]$$

and performing some operations between the first bracket and the third, it follows:
The relative difference between the two indices 
P_{0/1}^P\text{ and } L_{0/1}^P of type is equal to the ratio of brackets A'' and B'' C'' multiplied by parentheses where:

• the bracket A is the relative change in the quantity sold in the range “a”
  (relative to the average volume of all quantities’ change from the same group),

• the bracket B is the relative change in price assortment “a” (to change average of prices of all varieties of the same group).

The ratio of the two brackets is the formula of the coefficient of elasticity generated by the change of the quantities’ correlation based on the change of prices’ correlation (both correlation being defined for the product “a” to the same group). The elasticity coefficient obtained shows how the proportion or percent change the correlation between increasing the quantity of a product and the mean of all quantities in the same group when the proportion or correlation between price changes and the average variations of price for the same group changes by 1%:

\[ \lambda_{q/p}^a = \left[ \frac{\sum q_{i1}p_{i0}}{\sum q_{i0}p_{i0}} - 1 \right] : \left[ \frac{\sum p_{i1}q_{i0}}{\sum p_{i0}q_{i0}} - 1 \right] = \frac{p_{1/0}^P - L_{1/0}^P}{\omega(\sigma_{p_{1/0}})^2} \left( \frac{\sum p_{i1}q_{i0}}{\sum p_{i0}q_{i0}} \right)^2 \]
M.C. Demetrescu’s formula proposed allows the calculation of elasticity for any commodity “a” with the only restriction on the existence of a perfect correlation between \( q_{i1}' / q_{i0}' \) and \( p_{i1}' / p_{i0}' \) (restriction that do not exist across the same group).

If the measured difference is the existing one between CPI interpreter index formulas type (Paasche formula and Laspeyres formula) then the resulting elasticity for the commodity “a” is the elasticity of substitution of the specific good with the rest of all other commodities.

Finally, due to the dispersion of the price, difference \( \frac{p_{1/0}' - L_{1/0}'}{L_{1/0}'} \) is greater than or less than the extent that elasticity depending on the correlation of quantity – price interdependence is more or less. In terms of “caeteris paribus”, the higher the representative elasticity of substitution between the goods will be, as higher will be the relative difference between the interpreter indices.

Conclusions

The difference between two statistical space-time constructions called classical Index – Numbers (Laspeyres or Paasche) is to be found during such a mathematical demonstration in the essence of the Coefficient of elasticity. There still exist excellent resources in the statistical thinking specific to the Professor M.C. Demetrescu’s approach, which has neither beginning nor end, as statistics science remain for ever from its originality and creativity point of view...

Selective bibliography

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